Re-formulating the Einstein equations for stable numerical simulations

– Formulation Problem in Numerical Relativity –

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Abstract

We review recent efforts to re-formulate the Einstein equations for fully relativistic numerical simulations. The so-called numerical relativity (computational simulations in general relativity) is a promising research field matching with ongoing astrophysical observations such as gravitational wave astronomy. Many trials for longterm stable and accurate simulations of binary compact objects have revealed that mathematically equivalent sets of evolution equations show different numerical stability in free evolution schemes. In this article, we first review the efforts of the community, categorizing them into the following three directions: (1) modifications of the standard Arnowitt-Deser-Misner equations initiated by the Kyoto group, (2) rewriting of the evolution equations in hyperbolic form, and (3) construction of an "asymptotically constrained" system. We next introduce our idea for explaining these evolution behaviors in a unified way using eigenvalue analysis of the constraint propagation equations. The modifications of (or adjustments to) the evolution equations change the character of constraint propagation, and several particular adjustments using constraints are expected to diminish the constraint-violating modes. We propose several new adjusted evolution equations, and include some numerical demonstrations. We conclude by discussing some directions for future research.

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Contents

1 Overview

1.1 Numerical Relativity

The theory of general relativity describes the nature of the strong gravitational field. The Einstein equation predicts quite unexpected phenomena such as gravitational collapse, gravitational waves, the expanding universe and so on, which are all attractive not only for researchers but also for the public. The Einstein equation consists of 10 partial differential equations (elliptic and hyperbolic) for 10 metric components, and it is not easy to solve them for any particular situation. Over the decades, people have tried to study the general-relativistic world by finding its exact solutions, by developing approximation methods, or by simplifying the situations. Among these approaches, direct numerical integration of the Einstein equations can be said to be the most robust way to study the strong gravitational field. This research field is often called "numerical relativity".

Numerical Relativity **Box 1.1**

= Necessary for unveiling the nature of strong gravity. For example:

- gravitational waves from colliding black holes, neutron stars, supernovae, ...
- relativistic phenomena like cosmology, active galactic nuclei, ...
- mathematical feedback to singularity, exact solutions, chaotic behavior, ...
- laboratory for gravitational theories, higher-dimensional models, ...

Numerical relativity is now an essential field in gravity research. The current mainstream in numerical relativity is to analyze the final phase of compact binary objects (black holes and/or neutron stars) related to gravitational wave observations (see e.g. the conference proceedings [66]). Over the past decades, many groups have developed their numerical simulations by trial and error. Simulations require large-scale computational facilities, and long-time stable and accurate calculations. So far, we have achieved certain successes in simulating the coalescence of binary neutron stars (see e.g. [75]) and binary black holes (see e.g.[6]). However, people have still been faced with unreasonable numerical blow-ups at the end of simulations.

Difficulties in accurate/stable long-term evolution were supposed to be overcome by choosing proper gauge conditions and boundary conditions. However, recent several numerical experiments show that the (standard) Arnowitt-Deser-Misner (ADM) approach [12, 80, 98] is not the best formulation for numerics, and finding a better formulation has become one of the main research topics. A majority of workers in the field now believe in the existence of constraint-violating modes in most of the formulations. Thus, the stability problem is now shedding light on the mathematical structure of the Einstein equations.

The purpose of this article is to review the formulation problem in numerical relativity. Generally speaking, there are many open issues in numerical relativity, both theoretical (mathematical or physical) and numerical. We list major topics in Box 1.2. More general and recent introductions to numerical relativity are available, e.g. by d'Inverno (1996) [50], Seidel (1996/98/99) [73], Brügmann (2000) [25], Lehner (2001) [56], van Putten (2001) [88], and Baumgarte-Shapiro (2002) [16].

Numerical Relativity – open issues **Box 1.2**

0. How to select the foliation method of space-time
   Cauchy $(3+1)$, characteristic $(2+2)$, or combined?

$\Rightarrow$ if the foliation is $(3+1)$, then $\cdots$

1. How to prepare the initial data

   Theoretical: Proper formulation for solving constraints?
   How to prepare realistic initial data?
   Effects of background gravitational waves?
   Connection to the post-Newtonian approximation?

   Numerical: Techniques for solving coupled elliptic equations?
   Appropriate boundary conditions?

2. How to evolve the data

   Theoretical: Free evolution or constrained evolution?
   Proper formulation for the evolution equations? $\Leftarrow\Leftarrow\Leftarrow$ this review
   Suitable slicing conditions (gauge conditions)?

   Numerical: Techniques for solving the evolution equations?
   Appropriate boundary treatments?
   Singularity excision techniques?
   Matter and shock surface treatments?
   Parallelization of the code?

3. How to extract the physical information

   Theoretical: Gravitational wave extraction?
   Connection to other approximations?

   Numerical: Identification of black hole horizons?
   Visualization of simulations?

1.2 Formulation Problem in Numerical Relativity: Overview

There are several different approaches to simulating the Einstein equations. Among them the most robust way is to apply 3+1 (space + time) decomposition of space-time, as was first formulated by Arnowitt, Deser and Misner (ADM) [12] (we call this the "original ADM" system). [1]

If we divide the space-time into 3+1 dimensions, the Einstein equations form a constrained system: constraint equations and evolution equations. The system is quite similar to that of the Maxwell equations (Box 1.3),

[1] One alternative method of space-time foliation is the so-called characteristic approach $(2+2$ space-time decomposition). See reviews e.g. by d'Inverno (1996) [50], Winicour [89], Lehner (2001) [56]. Even in the 3+1 ADM approach, we concentrate the standard finite differential scheme to express numerical expression of space-time. See e.g. Brewin [23] for a recent progress in a lattice method.

The Maxwell equations : **Box 1.3**

The evolution equations: $(\partial_t = \partial/\partial t)$

$$\partial_t \mathbf{E} = \text{rot } \mathbf{B} - 4\pi \mathbf{j}, \qquad \text{and} \qquad \partial_t \mathbf{B} = -\text{rot } \mathbf{E} \tag{1.1}$$

Constraint equations:

$$\text{div } \mathbf{E} = 4\pi\rho, \qquad \text{and} \qquad \text{div } \mathbf{B} = 0 \tag{1.2}$$

where people solve constraint equations on the initial data, and use evolution equations to follow the dynamical behaviors.

In numerical relativity, this free-evolution approach is also the standard. This is because solving the constraints (non-linear elliptic equations) is numerically expensive, and because free evolution allows us to monitor the accuracy of numerical evolution. In black-hole treatments, recent "excision" techniques do not require one to impose explicit boundary conditions on the horizon, which is also a reason to apply free evolution scheme. As we will show in the next section, the standard ADM approach has two constraint equations; the Hamiltonian (or energy) and momentum constraints.

Up to a couple of years ago, the "standard ADM" decomposition [80, 98] of the Einstein equation was taken as the standard formulation for numerical relativists. However, numerical simulations were often interrupted by unexplained blow-ups (Figure.1). This was thought due to the lack of resolution, or inappropriate gauge choice, or the particular numerical scheme which was applied. However, after the accumulation of much experience, people have noticed the importance of the formulation of the evolution equations, since there are apparent differences in numerical stability although the equations are mathematically equivalent [2].

At this moment, there are three major ways to obtain longer time evolutions. Of course, the ideas, procedures, and problems are mingled with each other. The purpose of this article is to review all three approaches and to introduce our idea to view them in a unified way. Table 1 is a list of references.

(1) The first possibility is to use a modification of the ADM system developed by the Kyoto group [63, 64] (often cited as Shibata and Nakamura [74]) and later re-introduced by Baumgarte and Shapiro [15]. This is a combination of new variables, conformal decomposition, rescaling of the conformal factor, and replacement of terms in the evolution equation using momentum constraints (see §2.1).

(2) The second direction is to re-formulate the Einstein equations in a first-order hyperbolic form. This is motivated from the expectation that the symmetric hyperbolic system has well-posed properties in its Cauchy treatment in many systems and also that the boundary treatment can be improved if we know the characteristic speed of the system. In constructing hyperbolic systems, the essential procedures are to adjust equations using constraints and to introduce new variables, normally the spatially derivatived metric (see §2.2).

[2] The word *stability* is used quite different ways in the community.

- We mean by *numerical stability* a numerical simulation which continues without any blow-ups and in which data remains on the constrained surface.
- *Mathematical stability* is defined in terms of the well-posedness in the theory of partial differential equations, such that the norm of the variables is bounded by the initial data. See eq. (2.34) and around.
- For numerical treatments, there is also another notion of *stability*, the stability of finite differencing schemes. This means that numerical errors (truncation, round-off, etc) are not growing by evolution, and the evaluation is obtained by von Neumann's analysis. Lax's equivalence theorem says that if a numerical scheme is consistent (converging to the original equations in its continuum limit) and stable (no error growing), then the simulation represents the right (converging) solution. See [30] for the Einstein equations.

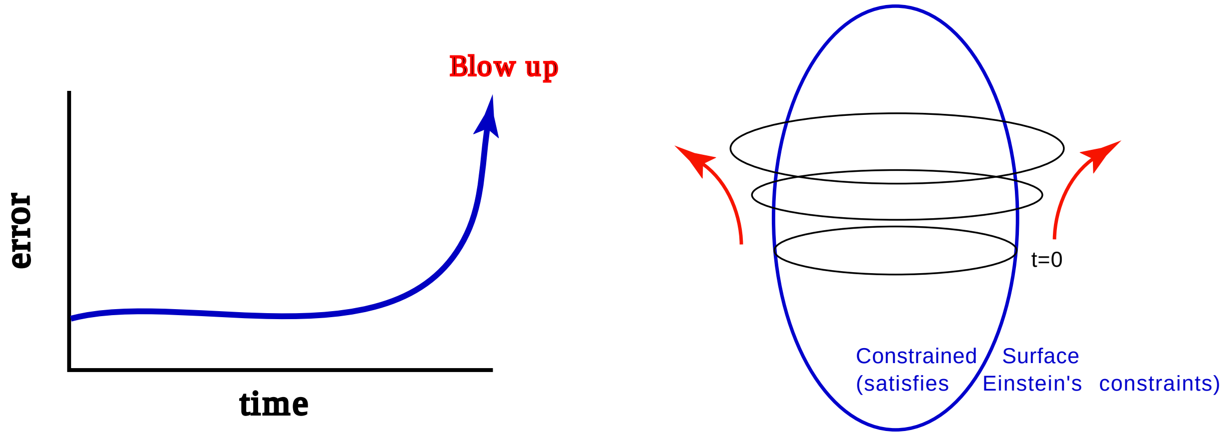

Figure 1: Origin of the problem for numerical relativists: Numerical evolutions depart from the constraint surface.

(3) The third is to construct a system which is robust against the violation of constraints, such that the constraint surface is an attractor. The idea was first proposed as a "λ-system" by Brodbeck et al [24] in which they introduce artificial flow to the constraint surface using a new variable based on the symmetric hyperbolic system (see §2.3).

The third idea has been generalized by us as an asymptotically constrained system. The main procedure is to adjust the evolution equations using the constraint equations [94, 95, 78]. The method is also applied to explain why the above approach (1) works, and also to propose alternative systems based on the ADM [95, 78] and BSSN [96] equations. Section 3 is devoted to explain this idea with an analytical tool of the eigenvalue analysis of the constraint propagation.

We follow the notations of that of MTW[61], i.e. the signature of the space-time is $(-+++)$, and the Riemann curvature is defined as

$$R^{\mu}{}_{\nu\alpha\beta} \equiv \partial_\alpha \Gamma^{\mu}_{\nu\beta} - \partial_\beta \Gamma^{\mu}_{\nu\alpha} + \Gamma^{\mu}{}_{\sigma\alpha}\Gamma^{\sigma}{}_{\nu\beta} - \Gamma^{\mu}{}_{\sigma\beta}\Gamma^{\sigma}{}_{\nu\alpha} \tag{1.3}$$

$$R_{\mu\nu} \equiv R^{\alpha}{}_{\mu\alpha\nu} \tag{1.4}$$

We use $\mu, \nu = 0, \cdots, 3$ and $i, j = 1, \cdots, 3$ as space-time indices. The unit $c = 1$ is applied. The discussion is mostly to the vacuum space-time, but the inclusion of matter is straightforward.

		formulations	numerical applications
(0) The standard ADM formulation			
	ADM	1962 Arnowitt-Deser-Misner [12, 80]	⇒ many
(1) The BSSN formulation			
	BSSN	1987 Nakamura et al [63, 64, 74]	⇒ 1987 Nakamura et al [63, 64]
			⇒ 1995 Shibata-Nakamura [74]
			⇒ 2002 Shibata-Uryu [75] etc
		1999 Baumgarte-Shapiro [15]	⇒ 1999 Baumgarte-Shapiro [15]
			⇒ 2000 Alcubierre et al [5, 7]
			⇒ 2001 Alcubierre et al [6] etc
		1999 Alcubierre et al [8]	
		1999 Frittelli-Reula [42]	
		2002 Laguna-Shoemaker [55]	⇒ 2002 Laguna-Shoemaker [55]
(2) The hyperbolic formulations			
	BM	1989 Bona-Massó [18, 19, 20]	⇒ 1995 Bona et al [20, 21, 22]
			⇒ 1997 Alcubierre, Massó [2, 4]
		1997 Bona et al [21]	⇒ 2002 Bardeen-Buchman [17]
		1999 Arbona et al [11]	
	CB-Y	1995 Choquet-Bruhat and York [32]	⇒ 1997 Scheel et al [70]
		1995 Abrahams et al [1]	⇒ 1998 Scheel et al [71]
		1999 Anderson-York [10]	⇒ 2002 Bardeen-Buchman [17]
	FR	1996 Frittelli-Reula [41]	⇒ 2000 Hern [44]
		1996 Stewart [81]	
	KST	2001 Kidder-Scheel-Teukolsky [52]	⇒ 2001 Kidder-Scheel-Teukolsky [52]
			⇒ 2002 Calabrese et al [27]
			⇒ 2002 Lindblom-Scheel [58]
		2002 Sarbach-Tiglio [69]	
	CFE	1981 Friedrich[36]	⇒ 1998 Frauendiener [35]
			⇒ 1999 Hübner [46]
	tetrad	1995 vanPutten-Eardley[86]	⇒ 1997 vanPutten [87]
	Ashtekar	1986 Ashtekar [13]	⇒ 2000 Shinkai-Yoneda [77]
		1997 Iriondo et al [48]	
		1999 Yoneda-Shinkai [92, 93]	⇒ 2000 Shinkai-Yoneda [77, 94]
(3) Asymptotically constrained formulations			
λ-system	to FR	1999 Brodbeck et al [24]	⇒ 2001 Siebel-Hübner [79]
	to Ashtekar	1999 Shinkai-Yoneda [76]	⇒ 2001 Yoneda-Shinkai [94]
adjusted	to ADM	1987 Detweiler [33]	⇒ 2001 Yoneda-Shinkai [95]
	to ADM	2001 Shinkai-Yoneda [95, 78]	⇒ 2002 Mexico NR Workshop [59]
	to BSSN	2002 Yoneda-Shinkai [96]	⇒ 2002 Mexico NR Workshop [59]
			⇒ 2002 Yo-Baumgarte-Shapiro [90]

Table 1: References to recent efforts of reformulating the Einstein equations. We list mainly those that have been applied to actual numerical comparisons.

2 The standard way and the three other roads

2.0 Strategy 0: The ADM formulation

2.0.1 The original ADM formulation

The idea of space-time evolution was first formulated by Arnowitt, Deser, and Misner (ADM) [12]. The formulation was first motivated by a desire to construct a canonical framework in general relativity, but it also gave the community to the fundamental idea of time evolution of space and time: such as foliations of 3-dimensional hypersurface (Figure 2). This *original* ADM formulation was translated to numerical relativists by Smarr [80] and York [98] in late 70s, with slightly different notations. We refer to the latter as the *standard* ADM formulation since this version is the starting point of the discussion.

The story begins by decomposing 4-dimensional space-time into 3 plus 1. The metric is expressed by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt), \tag{2.1}$$

where α and β_j are defined as $\alpha \equiv 1/\sqrt{-g^{00}}$ and $\beta_j \equiv g_{0j}$, and called the lapse function and shift vector, respectively. The projection operator or the intrinsic 3-metric g_{ij} is defined as $\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$, where $n_\mu = (-\alpha, 0, 0, 0)$ [and $n^\mu = g^{\mu\nu}n_\nu = (1/\alpha, -\beta^i/\alpha)$] is the unit normal vector of the spacelike hypersurface, Σ (see Figure 2). By introducing the extrinsic curvature,

$$K_{ij} \;=\; -\frac{1}{2}\pounds_n \gamma_{ij}, \tag{2.2}$$

and using the Gauss-Codacci relation, the Hamiltonian density of the Einstein equations can be written as

$$\mathcal{H}_{GR} \;=\; \pi^{ij}\dot\gamma_{ij} - \mathcal{L}, \quad \text{where} \quad \mathcal{L} = \sqrt{-g}R = \alpha\sqrt{\gamma}[{}^{(3)}R - K^2 + K_{ij}K^{ij}], \tag{2.3}$$

where π^{ij} is the canonically conjugate momentum to γ_{ij},

$$\pi^{ij} = \frac{\partial\mathcal{L}}{\partial\dot\gamma_{ij}} = -\sqrt{\gamma}(K^{ij} - K\gamma^{ij}), \tag{2.4}$$

omitting the boundary terms. The variation of $\mathcal{H}_{GR}$ with respect to α and β_i yields the constraints, and the dynamical equations are given by $\dot\gamma_{ij} = \frac{\delta\mathcal{H}_{GR}}{\delta\pi^{ij}}$ and $\dot\pi^{ij} = -\frac{\delta\mathcal{H}_{GR}}{\delta h_{ij}}$.

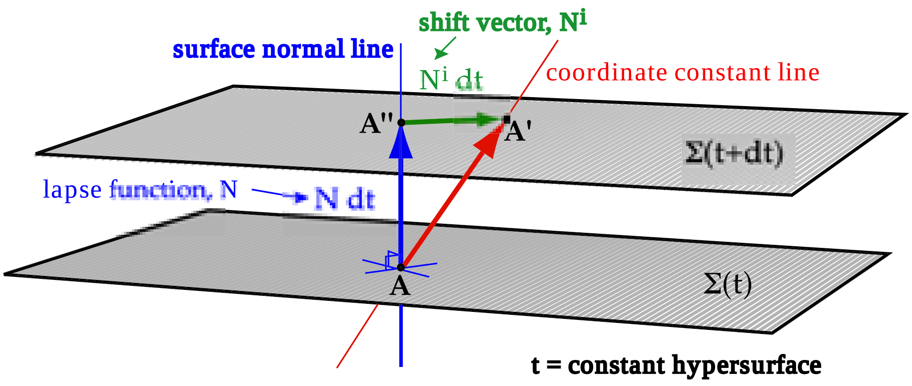

Figure 2: Concept of time evolution of space-time: foliations of 3-dimensional hypersurface. The lapse and shift functions are often denoted α or N, and β^i or N^i, respectively.

2.0.2 The standard ADM formulation

In the version of Smarr and York, K_{ij} was used as a fundamental variable instead of the conjugate momentum π^{ij} (see also the footnote [3]).

The Standard ADM formulation [80, 98]: **Box 2.1**

The fundamental dynamical variables are (γ_{ij}, K_{ij}), the three-metric and extrinsic curvature. The three-hypersurface Σ is foliated with gauge functions, (α, β^i), the lapse and shift vector.

- The evolution equations:

$$\begin{aligned} \partial_t \gamma_{ij} &= -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i, && (2.5) \\ \partial_t K_{ij} &= \alpha\,{}^{(3)}R_{ij} + \alpha K K_{ij} - 2\alpha K_{ik}K^k{}_j - D_iD_j\alpha \\ &\quad +(D_i\beta^k)K_{kj} + (D_j\beta^k)K_{ki} + \beta^k D_k K_{ij} && (2.6) \end{aligned}$$

  where $K = K^i{}_i$, and ${}^{(3)}R_{ij}$ and D_i denote three-dimensional Ricci curvature, and a covariant derivative on the three-surface, respectively.

- Constraint equations:

$$\begin{aligned} \mathcal{H}^{ADM} &:= {}^{(3)}R + K^2 - K_{ij}K^{ij} \approx 0, && (2.7) \\ \mathcal{M}_i^{ADM} &:= D_jK^j{}_i - D_iK \approx 0, && (2.8) \end{aligned}$$

  where ${}^{(3)}R = {}^{(3)}R^i{}_i$: these are called the Hamiltonian (or energy) and momentum constraint equations, respectively.

The formulation has 12 free first-order dynamical variables (γ_{ij}, K_{ij}), with 4 freedom of gauge choice (α, β_i) and with 4 constraint equations, (2.7) and (2.8). The rest freedom expresses 2 modes of gravitational waves.

The constraint propagation equations, which are the time evolution equations of the Hamiltonian constraint (2.7) and the momentum constraints (2.8), can be written as

The Constraint Propagations of the Standard ADM: **Box 2.2**

$$\begin{aligned} \partial_t \mathcal{H} &= \beta^j(\partial_j\mathcal{H}) + 2\alpha K\mathcal{H} - 2\alpha\gamma^{ij}(\partial_i\mathcal{M}_j) \\ &\quad +\alpha(\partial_l\gamma_{mk})(2\gamma^{ml}\gamma^{kj} - \gamma^{mk}\gamma^{lj})\mathcal{M}_j - 4\gamma^{ij}(\partial_j\alpha)\mathcal{M}_i, && (2.9) \\ \partial_t\mathcal{M}_i &= -(1/2)\alpha(\partial_i\mathcal{H}) - (\partial_i\alpha)\mathcal{H} + \beta^j(\partial_j\mathcal{M}_i) \\ &\quad +\alpha K\mathcal{M}_i - \beta^k\gamma^{jl}(\partial_i\gamma_{lk})\mathcal{M}_j + (\partial_i\beta_k)\gamma^{kj}\mathcal{M}_j. && (2.10) \end{aligned}$$

Further expressions of these constraint propagations are in Appendix A.

From these equations, we know that *if the constraints are satisfied on the initial slice Σ, then the constraints are satisfied throughout evolution.* The normal numerical scheme is to solve the elliptic

[3] We remark that there is one replacement in (2.6) using (2.7) in the process of conversion from the original ADM to the standard ADM equations. This is the key issue in the later discussion, and we shall be back this point in §3.2.

constraints for preparing the initial data, and to apply the free evolution (solving only the evolution equations). The constraints are used to monitor the accuracy of simulations.

2.0.3 Remarks

The ADM formulation was the standard formulation for numerical relativity up to the middle 90s. Numerous successful simulations were obtained for the problems of gravitational collapse, critical behavior, cosmology, and so on. However, stability problems have arisen for the simulations such as the gravitational radiation from compact binary coalescence, because the models require quite a long-term time evolution.

The origin of the problem was that the above statement in *Italics* is true in principle, but is not always true in numerical applications. A long history of trial and error began in the early 90s. From the next subsection we shall look back of them by summarizing "three strategies". We then unify these three roads as "adjusted systems", and as its by-product we show in §3.2 that the standard ADM equations *has* a constraint violating mode in its constraint propagation equations even for a single black-hole (Schwarzschild) spacetime [78].

2.1 Strategy 1: Modified ADM formulation by Nakamura et al (The BSSN formulation)

2.1.1 Introduction

Up to now, the most widely used formulation for large scale numerical simulations is a modified ADM system, which is now often cited as the Baumgarte-Shapiro-Shibata-Nakamura (BSSN) formulation. This reformulation was first introduced by Nakamura *et al.* [63, 64, 74]. The usefulness of this reformulation was re-introduced by Baumgarte and Shapiro [15], then was confirmed by other groups to show a long-term stable numerical evolution [5, 7]. The procedure is to apply conformally decomposition of the ADM variables and to implement their dynamical equations with several replacements.

2.1.2 Basic variables and equations

The widely used notation[15] introduces the variables $(\varphi, \tilde{\gamma}_{ij}, K, \tilde{A}_{ij}, \tilde{\Gamma}^i)$ instead of (γ_{ij}, K_{ij}), where

$$\begin{aligned}
\varphi &= (1/12)\log(\det\gamma_{ij}), && (2.11)\\
\tilde{\gamma}_{ij} &= e^{-4\varphi}\gamma_{ij}, && (2.12)\\
K &= \gamma^{ij}K_{ij}, && (2.13)\\
\tilde{A}_{ij} &= e^{-4\varphi}(K_{ij}-(1/3)\gamma_{ij}K), && (2.14)\\
\tilde{\Gamma}^i &= \tilde{\Gamma}^i_{jk}\tilde{\gamma}^{jk}. && (2.15)
\end{aligned}$$

The new variable $\tilde{\Gamma}^i$ was introduced in order to calculate Ricci curvature more accurately. $\tilde{\Gamma}^i$ also contributes to making system re-produce wave equations in its linear limit.

In BSSN formulation, Ricci curvature is not calculated as

$$R^{ADM}_{ij} = \partial_k\Gamma^k_{ij} - \partial_i\Gamma^k_{kj} + \Gamma^l_{ij}\Gamma^k_{lk} - \Gamma^l_{kj}\Gamma^k_{li}, \quad (2.16)$$

but

$$\begin{aligned}
R^{BSSN}_{ij} &= R^{\varphi}_{ij} + \tilde{R}_{ij}, && (2.17)\\
R^{\varphi}_{ij} &= -2\tilde{D}_i\tilde{D}_j\varphi - 2\tilde{\gamma}_{ij}\tilde{D}^k\tilde{D}_k\varphi + 4(\tilde{D}_i\varphi)(\tilde{D}_j\varphi) - 4\tilde{\gamma}_{ij}(\tilde{D}^k\varphi)(\tilde{D}_k\varphi), && (2.18)\\
\tilde{R}_{ij} &= -(1/2)\tilde{\gamma}^{lk}\partial_l\partial_k\tilde{\gamma}_{ij} + \tilde{\gamma}_{k(i}\partial_{j)}\tilde{\Gamma}^k + \tilde{\Gamma}^k\tilde{\Gamma}_{(ij)k} + 2\tilde{\gamma}^{lm}\tilde{\Gamma}^k_{l(i}\tilde{\Gamma}_{j)km} + \tilde{\gamma}^{lm}\tilde{\Gamma}^k_{im}\tilde{\Gamma}_{klj}, && (2.19)
\end{aligned}$$

where $\tilde{D}_i$ is covariant derivative associated with $\tilde{\gamma}_{ij}$. These are approximately equivalent, but R^{BSSN}_{ij} does have wave operator apparently in the flat background limit, so that we can expect more natural wave propagation behavior.

Additionally, the BSSN requires us to impose the conformal factor as

$$\tilde{\gamma}(:= \det\tilde{\gamma}_{ij}) = 1, \tag{2.20}$$

during evolution. This is a kind of definition, but can also be treated as a constraint.

The BSSN formulation [63, 64, 74, 15]: **Box 2.3**

The fundamental dynamical variables are $(\varphi, \tilde{\gamma}_{ij}, K, \tilde{A}_{ij}, \tilde{\Gamma}^i)$.

The three-hypersurface Σ is foliated with gauge functions, (α, β^i), the lapse and shift vector.

- The evolution equations:

$$\begin{aligned}
\partial_t^B \varphi &= -(1/6)\alpha K + (1/6)\beta^i(\partial_i\varphi) + (\partial_i\beta^i), && (2.21)\\
\partial_t^B \tilde{\gamma}_{ij} &= -2\alpha\tilde{A}_{ij} + \tilde{\gamma}_{ik}(\partial_j\beta^k) + \tilde{\gamma}_{jk}(\partial_i\beta^k) - (2/3)\tilde{\gamma}_{ij}(\partial_k\beta^k) + \beta^k(\partial_k\tilde{\gamma}_{ij}), && (2.22)\\
\partial_t^B K &= -D^iD_i\alpha + \alpha\tilde{A}_{ij}\tilde{A}^{ij} + (1/3)\alpha K^2 + \beta^i(\partial_i K), && (2.23)\\
\partial_t^B \tilde{A}_{ij} &= -e^{-4\varphi}(D_iD_j\alpha)^{TF} + e^{-4\varphi}\alpha(R^{BSSN}_{ij})^{TF} + \alpha K\tilde{A}_{ij} - 2\alpha\tilde{A}_{ik}\tilde{A}^k{}_j \\
&\quad +(\partial_i\beta^k)\tilde{A}_{kj} + (\partial_j\beta^k)\tilde{A}_{ki} - (2/3)(\partial_k\beta^k)\tilde{A}_{ij} + \beta^k(\partial_k\tilde{A}_{ij}), && (2.24)\\
\partial_t^B \tilde{\Gamma}^i &= -2(\partial_j\alpha)\tilde{A}^{ij} + 2\alpha(\tilde{\Gamma}^i_{jk}\tilde{A}^{kj} - (2/3)\tilde{\gamma}^{ij}(\partial_j K) + 6\tilde{A}^{ij}(\partial_j\varphi)) \\
&\quad -\partial_j(\beta^k(\partial_k\tilde{\gamma}^{ij}) - \tilde{\gamma}^{kj}(\partial_k\beta^i) - \tilde{\gamma}^{ki}(\partial_k\beta^j) + (2/3)\tilde{\gamma}^{ij}(\partial_k\beta^k)). && (2.25)
\end{aligned}$$

- Constraint equations:

$$\begin{aligned}
\mathcal{H}^{BSSN} &= R^{BSSN} + K^2 - K_{ij}K^{ij}, && (2.26)\\
\mathcal{M}_i^{BSSN} &= \mathcal{M}_i^{ADM}, && (2.27)\\
\mathcal{G}^i &= \tilde{\Gamma}^i - \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk}, && (2.28)\\
\mathcal{A} &= \tilde{A}_{ij}\tilde{\gamma}^{ij}, && (2.29)\\
\mathcal{S} &= \tilde{\gamma} - 1. && (2.30)
\end{aligned}$$

(2.26) and (2.27) are the Hamiltonian and momentum constraints (the "kinematic" constraints), while the latter three are "algebraic" constraints due to the requirements of BSSN formulation.

Hereafter we will write $\mathcal{H}^{BSSN}$ and $\mathcal{M}^{BSSN}$ simply as $\mathcal{H}$ and $\mathcal{M}$ respectively.

Taking careful account of these constraints, (2.26) and (2.27) can be expressed directly as

$$\begin{aligned}
\mathcal{H} &= e^{-4\varphi}\tilde{R} - 8e^{-4\varphi}\tilde{D}^j\tilde{D}_j\varphi - 8e^{-4\varphi}(\tilde{D}^j\varphi)(\tilde{D}_j\varphi) + (2/3)K^2 - \tilde{A}_{ij}\tilde{A}^{ij} - (2/3)\mathcal{A}K, && (2.31)\\
\mathcal{M}_i &= 6\tilde{A}^j{}_i(\tilde{D}_j\varphi) - 2\mathcal{A}(\tilde{D}_i\varphi) - (2/3)(\tilde{D}_iK) + \tilde{\gamma}^{kj}(\tilde{D}_j\tilde{A}_{ki}). && (2.32)
\end{aligned}$$

In summary, the fundamental dynamical variables in BSSN are $(\varphi, \tilde{\gamma}_{ij}, K, \tilde{A}_{ij}, \tilde{\Gamma}^i)$, 17 in all. The gauge quantities are (α, β^i) of which there are 4, and the constraints are $(\mathcal{H}, \mathcal{M}_i, \mathcal{G}^i, \mathcal{A}, \mathcal{S})$, i.e. 9 components. As a result, 4 (2 by 2) components are left which correspond to two gravitational polarization modes.

2.1.3 Remarks

Why BSSN is better than the standard ADM? Together with numerical comparisons with the standard ADM case[7, 57], this question has been studied by many groups using different approaches. Using numerical test evolution, Alcubierre et al [5] found that the essential improvement is in the process of replacing terms by the momentum constraints. They also pointed out that the eigenvalues of BSSN *evolution equations* have fewer "zero eigenvalues" than those of ADM, and they conjectured that the instability might be caused by these "zero eigenvalues". Miller [60] applied von Neumann's stability analysis to the plane wave propagation, and reported that BSSN has a wider range of parameters, which produces stable evolution. Analogical conformal decomposition of the Maxwell equations are also reported [53]. An effort was made to understand the advantage of BSSN from the point of hyperbolization of the equations in its linearized limit [5, 68]. These studies provide some support regarding the advantage of BSSN, while it is also shown an example of an ill-posed solution in BSSN (as well in ADM) by Frittelli and Gomez [40]. (Inspired by the BSSN's conformal decomposition, several related hyperbolic formulations have also been proposed [11, 42, 8].)

As we shall discuss in §3.3, the stability of the BSSN formulation is due not only to the introductions of new variables, but also to the replacement of terms in the evolution equations using the constraints. Further, we will show several additional adjustments to the BSSN equations which are expected to give us more stable numerical simulations.

Recently, Laguna and Shoemaker [55] modified the BSSN slightly, and found a great improvement in simulating a Schwarzschild black-hole.

The Laguna-Shoemaker version of BSSN [55]: **Box 2.4**

Modifications to BSSN are

- to introduce conformal scalings also to K, A_{ij} and N as
$$^{(LS)}\hat{K} = e^{6\,n\,\varphi}K, \qquad ^{(LS)}\hat{A}^i{}_j = e^{6\,n\,\varphi}A^i{}_j, \qquad N = e^{-6\,n\,\varphi}\alpha.$$
where $\hat{\gamma}_{ij} = e^{-4\varphi}\gamma_{ij}$ and n is a parameter ($n = 0$ recovers the BSSN variables)
- to use a mixed indices form $^{(LS)}\hat{A}^i{}_j$ rather than A_{ij}.
- to use a densitized lapse N rather than α.

There is no explicit explanation why these small changes work better than before, but we expect that our method can also be applied to finding the reason.

2.2 Strategy 2: Hyperbolic reformulations

2.2.1 Definitions, properties, mathematical backgrounds

The second effort to re-formulate the Einstein equations is to make the evolution equations reveal a first-order hyperbolic form explicitly. This is motivated by the expectation that the symmetric hyperbolic system has well-posed properties in its Cauchy treatment in many systems and also that the boundary treatment can be improved if we know the characteristic speed of the system. As a comprehensive review of the hyperbolic formulation, we refer to those by Choquet-Bruhat and York (1980) [31], Geroch (1996) [43], Reula (1998) [67], and Friedrich-Rendall (2000) [37],

We use the following definition:

Hyperbolic formulations **Box 2.5**

We say that the system is *a first-order (quasi-linear) partial differential equation system*, if a certain set of (complex-valued) variables u_α ($\alpha = 1, \cdots, n$) forms

$$\partial_t u_\alpha = \mathcal{M}^{l\beta}{}_\alpha(u)\, \partial_l u_\beta + \mathcal{N}_\alpha(u), \tag{2.33}$$

where $\mathcal{M}$ (the characteristic matrix) and $\mathcal{N}$ are functions of u but do not include any derivatives of u. Further we say the system is

- *a weakly hyperbolic system*, if all the eigenvalues of the characteristic matrix are real.
- *a strongly hyperbolic system* (or a diagonalizable / symmetrizable hyperbolic system), if the characteristic matrix is diagonalizable (has a complete set of eigenvectors) and has all real eigenvalues.
- *a symmetric hyperbolic system*, if the characteristic matrix is a Hermitian matrix.

We treat $\mathcal{M}^{l\beta}{}_\alpha$ as a $n \times n$ matrix when the l-index is fixed. The following properties of these matrices apply to every basis of l-index. We say λ^l is an eigenvalue of $\mathcal{M}^{l\beta}{}_\alpha$ when the characteristic equation, $\det(\mathcal{M}^{l\beta}{}_\alpha - \lambda^l \delta^\beta{}_\alpha) = 0$, is satisfied. The eigenvectors, p^α, are given by solving $\mathcal{M}^{l\alpha}{}_\beta\, p^{l\beta} = \lambda^l p^{l\alpha}$. The strong hyperbolicity is identified, e.g. by judging whether the multiplicity of its eigenvalue, n_λ, satisfies $\mathrm{rank}(\mathcal{M}^{l\beta}{}_\alpha - \lambda^l \delta^\beta{}_\alpha) = n - n_\lambda$ for all λ. In order to define the symmetric hyperbolic system, we need to declare an inner product $\langle u|u\rangle$ to determine whether $\mathcal{M}^{l\beta}{}_\alpha$ is Hermitian. In other words, we are required to define the way of lowering the index α of u^α. We say $\mathcal{M}^{l\beta}{}_\alpha$ is Hermitian with respect to this index rule, when $\mathcal{M}^l{}_{\beta\alpha} = \bar{\mathcal{M}}^l{}_{\alpha\beta}$ for every l, where the overhead bar denotes complex conjugate. To avoid this requirement of the definition of the inner product, people sometimes use the word *symmetrizable*, when the characteristic matrix becomes Hermitian by a certain *symmetrizer* (positive definite matrix). In our classification, this is only equivalent to the strongly hyperbolic system. [4]

Writing the system in a hyperbolic form is a quite useful step in proving that the system is *well-posed*. The mathematical well-posedness of the system means (1°) local existence (of at least one solution u), (2°) uniqueness (i.e., at most solutions), and (3°) stability (or continuous dependence of solutions $\{u\}$ on the Cauchy data) of the solutions. The resultant statement expresses the existence of the energy inequality on its norm,

$$||u(t)|| \leq e^{\alpha\tau}||u(t=0)||, \qquad \text{where } 0 < \tau < t, \quad \alpha = const. \tag{2.34}$$

This indicates that the norm of $u(t)$ is bounded by a certain function and the initial norm. Remark that this mathematical boundness does not mean that the norm $u(t)$ decreases along the time evolution.

The inclusion relation of the hyperbolicities is,

$$\text{symmetric hyperbolic} \subset \text{strongly hyperbolic} \subset \text{weakly hyperbolic}. \tag{2.35}$$

The Cauchy problem under weak hyperbolicity is not, in general, C^∞ well-posed. At the strongly hyperbolic level, we can prove the finiteness of the energy norm if the characteristic matrix is independent of u (cf [81]), that is one step definitely advanced over a weakly hyperbolic form. Similarly, the

[4] Several groups use a slightly different definition for a symmetric hyperbolic system: defining when the principal symbol of the system $i\mathcal{M}^{l\beta}{}_\alpha k_l$ is anti-Hermitian for arbitrary vector k_l. The two definitions are equivalent when the vector k_l is real-valued, but different eigenvalues. In our definition, all eigenvalues are real-valued, while in the other they are all pure imaginary.

well-posedness of the symmetric hyperbolic is guaranteed if the characteristic matrix is independent of u, while if it depends on u we have only limited proofs for the well-posedness. From the mathematical point of view, proving well-posedness with less strict conditions is an old but still active research problem. Therefore we have to recall that even if we construct a symmetric hyperbolic system in general relativity, that equation does not necessarily guarantee numerical stability.

From the point of numerical applications, to hyperbolize the evolution equations is quite attractive, not only for its mathematically well-posed features. The expected additional advantages are the following.

(a) It is well known that a certain flux conservative hyperbolic system is taken as an essential formulation in the computational Newtonian hydrodynamics when we control shock wave formations due to matter (e.g. [45]).

(b) The characteristic speed (eigenvalues of the principal matrix) is supposed to be the propagation speed of the information in that system. Therefore it is naturally imagined that these magnitudes are equivalent to the physical information speed of the model to be simulated.

(c) The existence of the characteristic speed of the system is expected to give us an improved treatment of the numerical boundary, and/or to give us a new well-defined Cauchy problem within a finite region (the so-called initial boundary value problem, IBVP).

These statements sound reasonable, but have not yet been generally confirmed in actual numerical simulations. But we are safe in saying that the formulations are not yet well developed to test these issues. For example, IBVP studies are preliminary yet, and most works are based on a particular symmetric hyperbolic and in a limited space-time symmetry [26, 29, 38, 49, 81, 82, 83, 84]. We will come back to this issue in §2.2.3 or §4, but meanwhile let us view the hyperbolic formulations from the comparisons of pure evolution equations.

We note that rich mathematical theories on partial differencial equations are obtained in a first-order form, while there is a study on linearized ADM equations in a second-order form [54].

2.2.2 Hyperbolic formulations of the Einstein equations

As was discussed by Geroch [43], most physical systems can be expressed as symmetric hyperbolic systems. In order to prove that the Einstein's theory is a well-posed system, to hyperbolize the Einstein equations is a long-standing research area in mathematical relativity. As we mentioned in the introduction, numerical relativity shed light on this mathematical problem.

The standard ADM system does not form a first order hyperbolic system. This can be seen immediately from the fact that the ADM evolution equation (2.6) has Ricci curvature in RHS. So far, several first order hyperbolic systems of the Einstein equation have been proposed. In constructing hyperbolic systems, the essential procedures are (1°) to introduce new variables, normally the spatially derivatived metric, (2°) to adjust equations using constraints. Occasionally, (3°) to restrict the gauge conditions, and/or (4°) to rescale some variables. Due to process (1°), the number of fundamental dynamical variables is always larger than that of ADM.

In the following discussion, we briefly review several hyperbolic systems of the Einstein equations. We only mention the systems which applied numerical comparisons. See Table. 1 for a more comprehensive list.

- The Bona-Massó formulation [18, 19, 20, 21]
  This was the first active effort to apply hyperbolized equations to numerical relativity. They introduced auxiliary variables to reduce the system first order in space: $A_k = \partial_k \ln \alpha$, $\quad B_k{}^i = (1/2)\partial_k \beta^i$, and $D_{kij} = (1/2)\partial_k g_{ij}$, and construct a first order flux conservative system. In their latest formulation [21], the set of dynamical variables is $(\alpha, \gamma_{ij}, K_{ij}, A_i, D_{rij}, V_i \equiv 2D_{[ir]}{}^r)$, 37

functions in total, and the lapse function is restricted to a certain functional condition. The system is a symmetrizable hyperbolic. They observed improved numerical stability compared to that of the standard ADM system in spherical symmetric spacetime evolution [20]. An advantage of having characteristic speed is also applied to improve the treatment of the outer boundary condition [22]. However, the appearance of shock formation was also reported unless the lapse function is determined by solving the elliptic or parabolic equations [2, 4].

- The Einstein-Ricci system [32, 1] / Einstein-Bianchi system [9]
  Series of works by Choquet-Bruhat, York, and their colleagues developed hyperbolic systems in slightly different ways. They intended to construct wave-type equations ($\Box\phi = S$) for (γ_{ij}, K_{ij}), that require the introduction of new variables (45 or 66 variables in total) and the use of Bianchi identities. The resultant system is a third-order system in time for γ_{ij} since they use the equation $\partial_t R_{ij}$, but has only physical characteristic speed (zero or light speed). Scheel et al [70, 71] developed a numerial code with this formulation, and reported that they could evolve Schwarzschild black hole (with 1-dimensional code) quite long time ($\sim 10^4 M$), but futher modifications to equations were necesarrily.

- The Einstein-Christoffel system [10]
  Anderson and York [10] constructed a symmetrizable hyperbolic system which only has physical characteristic speed. The variables are fundamentally $(\gamma_{ij}, K_{ij}, \Gamma_{kij})$, and they also derived a set of equations with $(\gamma_{ij}, K_{ij}, G_{kij} \equiv \partial_k \gamma_{ij})$, 30 functions. Their formulation differs from the Bona-Massó formulation in the way the momentum constraint is used to make the system hyperbolic. Using a model of plane wave propagation, Bardeen and Buchman [17] numerically compared this formulation with ADM and Bona-Massó with slight changes in variables. Their experiments are of plane-symmetric wave propagations, and they studies the boundary treatment in detail. We will mention their work later.

- The Ashtekar formulation [13]
  Ashtekar's reformulation of space-time was introduced to provide a new non-perturbative approach to quantum gravity. The new basic variables are the densitized inverse triad, $\tilde{E}^i_a$, and the SO(3,C) self-dual connection, $\mathcal{A}^a_i$, where the indices $i, j, \cdots$ indicate the 3-spacetime, and $a, b, \cdots$ are for SO(3) space. The formulation requires additional constraints and reality conditions in order to describe the classical Lorentzian space-time evolution, but the evolution system itself forms a weakly hyperbolic system. By keeping the number of variables the same [$(\tilde{E}^i_a, \mathcal{A}^a_i)$=total 18 (minimum ever)], we can construct both strongly and symmetric hyperbolic systems by adjusting equations with constraints and/or restricting gauge conditions [92, 93]. The authors made numerical comparisons between the hyperbolicity of the systems with plane symmetric gravitational wave propagation using a periodic boundary condition [77, 94]. The outcome is that there are no drastic differences in numerical stability between the three levels of the hyperbolic equations. We will describe the details in Appendix B.

- The Frittelli-Reula formulation [41, 81]
  This is a one-parameter family of a symmetric hyperbolic system. The procedure is to define new variables, to adjust evolution equations with constraints (with one parameter), and to densitize the lapse function (with one parameter). The variables are $(\gamma_{ij}, M^{ij}{}_k \equiv \frac{1}{2}(\gamma^{ij}{}_{,k} + a\gamma^{ij}\gamma_{rs}\gamma^{rs}{}_{,k}), P^{ij} \equiv K^{ij} + b\gamma^{ij}K)$ where a, b are two other parameters, and make a total of 30. They define a symmetric hyperbolic system by non-diverging energy norm, and restricted free parameter spaces. A numerical comparison was also made for Gowdy spacetime [44], with quite similar conclusions as in the Ashtekar version.

- The Conformal Field equations [36]
  A series of works by Friedrich [36] attempted to construct a $3+1$ formulation with hyperboloidal

foliations (i.e. asymptotically null foliations), and with comformal compactification. This is the ultimate plan to remove the outer boundary problem in numerical simulation, and to provide a suitable foliation for gravitational radiation problem. However, the current equations are rather quite complicated. In its metric-based expression [46], the evolution variables are 57; γ_{ij}, K_{ij}, the connection coefficients $\gamma^a{}_{bc}$, projections ${}^{(0,1)}\hat{R}_a = n^b\gamma_a{}^c\hat{R}_{bc}$ and ${}^{(1,1)}\hat{R}_{ab} = \gamma_a{}^c\gamma_b{}^d\hat{R}_{bd}$ of 4-dimensional Ricci tensor $\hat{R}_{ab}$, the electric and magnetic components of the rescaled Weyl tensor $C_{abc}{}^d$, and the comformal factor Ω and its related quantities $\Omega_0 \equiv n^a\nabla_a\Omega, \nabla_a\Omega, \nabla^a\nabla_a\Omega$. By specifying suitable gauge functions (α, β^a, R) where R is the Ricci scalar, then the total system forms a symmetric hyperbolic system. Applications to numerical relativity are in progress, but have not yet reached the stage of applying evolution in a non-trivial metric. For more details, see reviews e.g. by Frauendiener [34] or by Husa [47].

- The Kidder-Scheel-Teukolsky (KST) formulation [52]
  This set of equations is a generalized formulation of the previous ones. It has 12 free parameters, and includes both Anderson-York[10] and Frittelli-Reula[41] formulations as a subset. Therefore it might be useful to discuss further hyperbolization methods starting with this KST formulation. We briefly summarize it in Box 2.6.

The Kidder-Scheel-Teukolsky (KST) formulation [52]: **Box 2.6**

- Starting from a set of $(g_{ij}, K_{ij}, d_{kij} \equiv \partial_k g_{ij})$, the fundamental dynamical variables are defined $(g_{ij}, P_{ij}, M_{kij})$, as

$$P_{ij} \equiv K_{ij} + \hat{z}g_{ij}K, \tag{2.36}$$

$$M_{kij} \equiv (1/2)[\hat{k}d_{kij} + \hat{e}d_{(ij)k} + g_{ij}(\hat{a}d_k + \hat{b}b_k) + g_{k(i}(\hat{c}d_{j)} + \hat{d}b_{j)})], \tag{2.37}$$

  where $d_k = g^{ab}d_{kab}$, $b_k = g^{ab}d_{abk}$, and $(\hat{a}, \hat{b}, \hat{c}, \hat{d}, \hat{e}, \hat{k}, \hat{z})$ are "kinematical" parameters.

- The 3-hypersurface Σ is foliated with gauge functions, (α, β^i), the lapse and shift vector. However, the densitized lapse, $Q = \log(\alpha g^{-\sigma})$, (with a parameter σ) is actually used.

- The evolution equations are adjusted with constraints [in a version of $(g_{ij}, K_{ij}, d_{kij})$]

$$\hat{\partial}_0 g_{ij} = -2\alpha K_{ij}, \tag{2.38}$$

$$\hat{\partial}_0 K_{ij} = (\cdots) + \gamma\alpha g_{ij}\mathcal{H} + \zeta\alpha g^{ab}\mathcal{C}_{a(ij)b}, \tag{2.39}$$

$$\hat{\partial}_0 d_{kij} = (\cdots) + \eta\alpha g_{k(i}\mathcal{M}_{j)} + \chi\alpha g_{ij}\mathcal{M}_k, \tag{2.40}$$

  where $\hat{\partial}_0 = \partial_t - \pounds_\beta$ and $(\gamma, \zeta, \eta, \chi)$ are parameters. The terms $(\cdots)$ are original terms derived from the ADM equations and are available as (2.14) and (2.24) in [52].

- Constraints are $(\mathcal{H}, \mathcal{M}_i, \mathcal{C}_{klij})$, where $\mathcal{C}_{klij} \equiv \partial_{[k}d_{l]ij}$.

In short, the number of dynamical variables and constraints are 30 and 22 and there are 12 free parameters. Although there is no specific method to specify 12 free parameters KST showed numerical examples of the Schwarzschild black hole (mass M) evolution, which runs quite a longer time ($t \sim 6000M$) [52, 72].

We think the essential advantage of the KST system is the introduction of "kinematical" parameters. These 6 parameters

- do not change the eigenvalues of evolution eqs.,
- do not affect the principal part of the constraint evolution eqs.,
- do affect the eigenvectors of the evolution system, and
- do affect the nonlinear terms of evolution eqs/constraint evolution eqs.

As Calabrese et al [27] pointed out, KST equations at linearized level on the flat spacetime have no non-principal terms, and these "kinematical" parameters finally enable us to discuss the features of hyperbolicity in numerical experiments. Several variations of the KST formulation are presented in [69]. Recently, Lindblom and Scheel [58] tried to explain the relation between the numerical error growing rate and the predicted error growing rate from the characteristic matrix of KST evolution equations. Their trial did not succeed in matching the two completely, but at least began to reveal a similar order before non-linear blow-up begins.

2.2.3 Remarks

When we discuss hyperbolic systems in the context of numerical stability, the following questions should be considered:

Questions to hyperbolic formulations on its applications to numerics

Q(A) From the point of the set of evolution equations, does hyperbolization actually contribute to numerical accuracy and stability? Under what conditions/situations will the advantages of hyperbolic formulation be observed?

Q(B) If the answer to Q(A) is affirmative, which level of hyperbolicity is practically useful for numerical applications? Strongly hyperbolic, symmetric hyperbolic, or other?

Q(C) If the answer to Q(A) is negative, then can we find other practical advantages to hyperbolization? Treatment of boundary conditions, or other?

Unfortunately, we do not have conclusive answers to these questions, but many experiences are being accumulated. Several earlier numerical comparisons reported the stability of hyperbolic formulations [20, 21, 22, 70, 71]. But we have to remember that this statement went against the standard ADM formulation, which has a constraint-violating mode for Schwarzschild spacetime as has been shown recently (see §3.2).

These partial numerical successes encouraged the community to formulate various hyperbolic systems. Recently, Calabrese et al [28] reported there is a certain differences in the long-term convergence features between weakly and strongly hyperbolic systems on the Minkowskii background space-time. However, several numerical experiments also indicate that this direction is not a complete success.

Objections from numerical experiments

- Above earlier numerical successes were also terminated with blow-ups.
- If the gauge functions are evolved according to the hyperbolic equations, then their finite propagation speeds may cause pathological shock formations in simulations [2, 4].
- There are no drastic differences in the evolution properties *between* hyperbolic systems (weakly, strongly and symmetric hyperbolicity) by systematic numerical studies by Hern [44] based on Frittelli-Reula formulation [41], and by the authors [77] based on Ashtekar's formulation [13, 92, 93].
- Proposed symmetric hyperbolic systems were not always the best ones for numerical evolution. People are normally still required to reformulate them for suitable evolution. Such

efforts are seen in the applications of the Einstein-Ricci system [71], the Einstein-Christoffel system [17], the conformal field equations [47], and so on.

- Bardeen and Buchmann [17] confirmed the usefulness of hyperbolicity when they treated numerical boundary conditions in plane-symmetric wave propagation problem, but they also mentioned that the same techniques can not be applied in 2 or 3-dimensional cases.

Of course, these statements only casted on a particular formulation, and therefore we have to be careful not to over-emphasize the results. In order to figure out the reasons for the above objections, it is worth stating the following cautions:

Remarks on hyperbolic formulations

(a) Rigorous mathematical proofs of well-posedness of PDE are mostly for simple symmetric or strongly hyperbolic systems. If the matrix components or coefficients depend on dynamical variables (as in all any versions of hyperbolized Einstein equations), almost nothing was proved in more general situations.

(b) The statement of "stability" in the discussion of well-posedness refers to the bounded growth of the norm, and does not indicate a decay of the norm in time evolution.

(c) The discussion of hyperbolicity only uses the characteristic part of the evolution equations, and ignores the rest.

We think the origin of confusion in the community results from over-expectation on the above issues. Mostly, point (c) is the biggest problem, as was already pointed out in several places [94, 56, 78]. The above numerical claims from Ashtekar and Frittelli-Reula formulations were mostly due to the contribution (or interposition) of non-principal parts in evolution.

Regarding this issue, the recent KST formulation finally opens the door. As we saw, KST's "kinematic" parameters enable us to reduce the non-principal part, so that numerical experiments are hopefully expected to represent predicted evolution features from PDE theories. At this moment, the agreement between numerical behavior and theoretical prediction is not yet perfect but close [58]. If further studies reveal the direct correspondences between theories and numerical results, then the direction of hyperbolization will remain as the essential approach in numerical relativity, and the related IBVP researches will become a main research subject in the future.

Meanwhile, it will be useful if we have an alternative procedure to predict stability including the effects of the non-principal parts of the equations, which are neglected in the discussion of hyperbolicity. Our proposal of adjusted system in the next subsection may be one of them.

2.3 Strategy 3: Asymptotically constrained systems

The third strategy is to construct a robust system against the violation of constraints, such that the constraint surface is an attractor (Figure 3). The idea was first proposed as "λ-system" by Brodbeck et al [24], and then developed in more general situations as "adjusted system" by the authors [94].

2.3.1 The "λ-system"

Brodbeck et al [24] proposed a system which has additional variables λ that obey artificial dissipative equations. The variable λs are supposed to indicate the violation of constraints and the target of the system is to get $\lambda = 0$ as its attractor. Their proposal can be summarized as in Box 2.7.

The "λ-system" (Brodbeck-Frittelli-Hübner-Reula) [24]: **Box 2.7**

For a symmetric hyperbolic system, add additional variables λ and artificial force to reduce the violation of constraints.

The procedure:

1. Prepare a symmetric hyperbolic evolution system $\quad \partial_t u = M \partial_i u + N$
2. Introduce λ as an indicator of violation of constraint which obeys dissipative eqs. of motion $\quad \partial_t \lambda = \alpha C - \beta \lambda$ $\quad (\alpha \neq 0, \beta > 0)$
3. Take a set of (u, λ) as dynamical variables $\quad \partial_t \begin{pmatrix} u \\ \lambda \end{pmatrix} \simeq \begin{pmatrix} A & 0 \\ F & 0 \end{pmatrix} \partial_i \begin{pmatrix} u \\ \lambda \end{pmatrix}$
4. Modify evolution eqs so as to form a symmetric hyperbolic system $\quad \partial_t \begin{pmatrix} u \\ \lambda \end{pmatrix} = \begin{pmatrix} A & \bar{F} \\ F & 0 \end{pmatrix} \partial_i \begin{pmatrix} u \\ \lambda \end{pmatrix}$

The main ideas are to introduce additional variables, λs, to impose dissipative dynamical equations for λs, and to construct a symmetric hyperbolic system for both original variables and λs. Since the total system is designed to have symmetric hyperbolicity, the evolution is supposed to be unique. Brodbeck et al showed analytically that such a decay of λs can be seen for sufficiently small $\lambda (> 0)$ with a choice of appropriate combinations of αs and βs.

Brodbeck et al presented a set of equations based on Frittelli-Reula's symmetric hyperbolic formulation [41]. The version of Ashtekar's variables was presented by the authors [76] for controlling the constraints or reality conditions or both (see §B.2.2). The numerical tests of both the Maxwell-λ-system and the Ashtekar-λ-system were performed [94], and confirmed to work as expected (see §B.3.3). Although it is questionable whether the recovered solution is true evolution or not [79], we think the idea is quite attractive. To enforce the decay of errors in its initial perturbative stage seems the key to the next improvements, which are also developed in the next section on "adjusted systems".

However, there is a high price to pay for constructing a λ-system. The λ-system can not be introduced generally, because (i) the construction of λ-system requires the original evolution equations to have a symmetric hyperbolic form, which is quite restrictive for the Einstein equations, (ii) the final system requires many additional variables and we also need to evaluate all the constraint equations at every time step, which is a hard task in computation. Moreover, (iii) it is not clear that the λ-system is robust enough for non-linear violation of constraints, or that λ-system can control constraints which

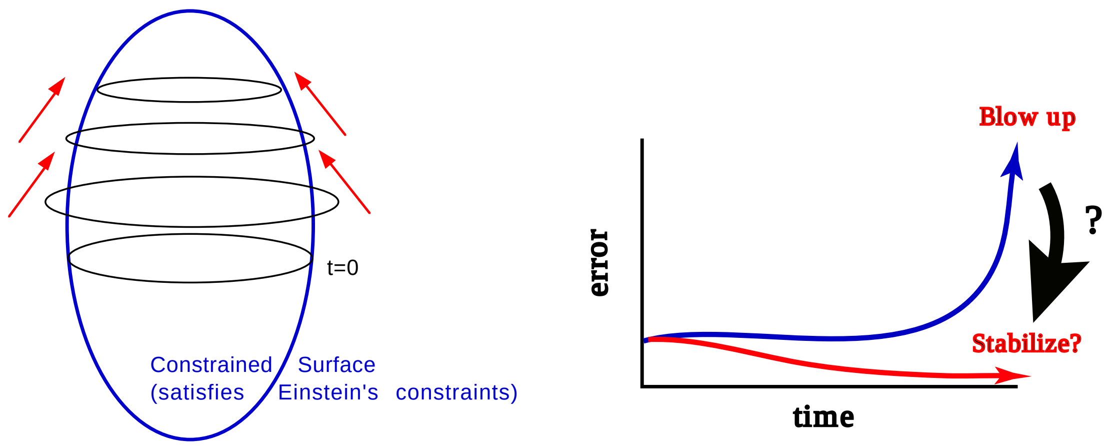

Figure 3: Schematic picture of "asymptotically constrained system".

do not have any spatial differential terms.

2.3.2 The "adjusted system"

Next, we propose an alternative system which also tries to control the violation of constraint equations actively, which we named "adjusted system". We think that this system is more practical and robust than the previous λ-system.

The Adjusted system (essentials) [94]: **Box 2.8**

Purpose:	Control the violation of constraints by reformulating the system so as to have a constrained surface as attractor.
Procedure:	Add a particular combination of constraints to the evolution equations, and adjust its multipliers.
Theoretical support:	Eigenvalue analysis of the constraint propagation equations.
Advantages:	Available even if the base system is not symmetric hyperbolic.
Advantages:	Keeps the number of the variables the same as in the original system.

We will describe the details in the next section, but summarize the procedures in advance:

The Adjusted system (procedures): **Box 2.9**

1.	Prepare a set of evolution eqs.	$\partial_t u = J\partial_i u + K$
2.	Add constraints in RHS	$\partial_t u = J\partial_i u + K \underbrace{+\kappa C}$
3.	Choose the coeff. κ so as to make the eigenvalues of the homogenized adjusted $\partial_t C$ eqs negative reals or pure imaginary. (See Box 3.2 and 3.3)	$\partial_t C = D\partial_i C + EC$ $\partial_t C = D\partial_i C + EC \underbrace{+F\partial_i C + GC}$

The details are in §3.

The process of adjusting equations is a common technique in other re-formulating efforts as we reviewed. However, we try to employ the evaluation process of constraint amplification factors as an alternative guideline to hyperbolization of the system.

For the Maxwell equation and the Ashtekar version of the Einstein equations, we numerically found that this idea works to reduce the violation of constraints, and that the effects are much better than by constructing its symmetric hyperbolic versions [94] (see §B.3.4). The idea was applied to the standard ADM formulation which is not hyperbolic and several attractive adjustments were proposed [95, 78] (see §3.2). This analysis was also applied to explain the advantages of the BSSN formulation, and again several alternative adjustments to BSSN equations were proposed [96] (see §3.3). We will explain these issues in the next section.

3 A unified treatment: Adjusted System

This section is devoted to present our idea of "asymptotically constrained system", which was briefly introduced in Box 2.8 and 2.9 in the previous section. We begin with an overview of the adjusting procedure and the idea of background structure in §3.1. Next, we show the applications both to ADM (§3.2) and BSSN (§3.3) formulations. Original references can be found in [94, 95, 78, 96].

3.1 Procedures : Constraint propagation equations and Proposals

Suppose we have a set of dynamical variables $u^a(x^i, t)$, and their evolution equations,

$$\partial_t u^a = f(u^a, \partial_i u^a, \cdots), \tag{3.1}$$

and the (first class) constraints,

$$C^\alpha(u^a, \partial_i u^a, \cdots) \approx 0. \tag{3.2}$$

Note that we do not require (3.1) forms a first order hyperbolic form. We propose to investigate the evolution equation of C^α (constraint propagation),

$$\partial_t C^\alpha = g(C^\alpha, \partial_i C^\alpha, \cdots), \tag{3.3}$$

for predicting the violation behavior of constraints in time evolution. We do not mean to integrate (3.3) numerically together with the original evolution equations (3.1), but mean to evaluate them analytically in advance in order to reformulate the equations (3.1).

There may be two major analyses of (3.3); (a) the hyperbolicity of (3.3) when (3.3) is a first order system, and (b) the eigenvalue analysis of the whole RHS in (3.3) after a suitable homogenization.

(a) Hyperbolicity analysis of (3.3)
If (3.3) forms a first-order system, the standard hyperbolic PDE analysis is applicable. As we viewed in §2.2, the analysis is mainly to identify the level of hyperbolicity and to calculate the characteristic speed of the system, from eigenvalues of the principal matrix.

For example, the evolution equations of the ADM formulations, (3.1) [(2.5) and (2.6)], do not form a first-order system, while their constraint propagation equations, (3.3) [(2.9) and (2.10)], do form. Therefore, we can apply the classification on the hyperbolicity (weakly, strongly or symmetric) to the constraint propagation equations. However, if one adjusts the ADM equations with constraints, then this first-order characters will not be guaranteed.

As we mentioned in §2.2.3, another big problem in the hyperbolic analysis is that it only discusses the principal part of the system. If there is a method to characterize the non-principal part, then that will help to clarify our understanding of evolution behavior.

(b) Eigenvalue analysis of the whole RHS in (3.3) after a suitable homogenizing procedure.
This analysis may compensate for the above problems.

Amplification Factors of Constraint Propagation equations: **Box 3.1**

We propose to homogenize (3.3) by a Fourier transformation, e.g.

$$\begin{aligned} \partial_t \hat{C}^\alpha &= \hat{g}(\hat{C}^\alpha) = M^\alpha{}_\beta \hat{C}^\beta, \\ &\qquad \text{where } C(x,t)^\rho = \int \hat{C}(k,t)^\rho \exp(ik\cdot x) d^3k, \end{aligned} \tag{3.4}$$

then to analyze the set of eigenvalues, say Λs, of the coefficient matrix, $M^\alpha{}_\beta$, in (3.4). We call Λs the constraint amplification factors (CAFs) of (3.3).

The CAFs predict the evolution of constraint violations. We therefore can discuss the "distance" to the constraint surface using the "norm" or "compactness" of the constraint violations (although we do not have exact definitions of these "$\cdots$" words).

The next conjecture seems to be quite useful to predict the evolution feature of constraints:

Conjecture on Constraint Amplification Factors (CAFs): **Box 3.2**

(A) If CAF has a *negative real-part* (the constraints are forced to be diminished), then we see more stable evolution than a system which has positive CAF.

(B) If CAF has a *non-zero imaginary-part* (the constraints are propagating away), then we see more stable evolution than a system which has zero CAF.

We found that the system becomes more stable when more Λs satisfy the above criteria. (The first observations were in the Maxwell and Ashtekar formulations [77, 94]). Actually, supporting mathematical proofs are available when we classify the fate of constraint propagations as follows.

Classification of Constraint propagations: **Box 3.3**

If we assume that avoiding the divergence of constraint norm is related to the numerical stability, the next classification would be useful:

(C1) *Asymptotically constrained :* All the constraints decay and converge to zero.
This case can be obtained if and only if all the real parts of CAFs are negative.

(C2) *Asymptotically bounded :* All the constraints are bounded at a certain value. (this includes the above *asymptotically constrained* case.)
This case can be obtained if and only if (a) all the real parts of CAFs are not positive and the constraint propagation matrix $M^\alpha{}_\beta$ is diagonalizable, or (b) all the real parts of CAFs are not positive and the real part of the degenerated CAFs is not zero.

(C3) *Diverge :* At least one constraint will diverge.

The details are shown in [97].

This classification roughly indicates the heuristic statements (A) in Box 3.2. A practical procedure for this classification is drawn in Figure 4.

We remark that this eigenvalue analysis requires the fixing of a particular background spacetime, since the CAFs depend on the dynamical variables, u^a.

The above features of the constraint propagation, (3.3), will differ when we modify the original evolution equations. Suppose we add (adjust) the evolution equations using constraints

$$\partial_t u^a = f(u^a, \partial_i u^a, \cdots) + F(C^\alpha, \partial_i C^\alpha, \cdots), \tag{3.5}$$

then (3.3) will also be modified as

$$\partial_t C^\alpha = g(C^\alpha, \partial_i C^\alpha, \cdots) + G(C^\alpha, \partial_i C^\alpha, \cdots). \tag{3.6}$$

Therefore, the problem is how to adjust the evolution equations so that their constraint propagations satisfy the above criteria as much as possible.

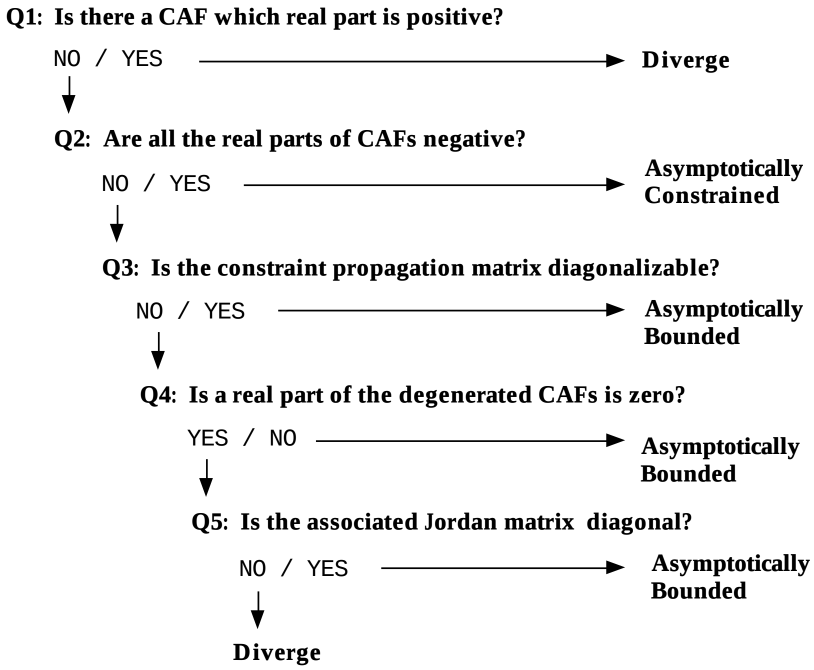

Figure 4: A flowchart to classify the constraint propagations (Box 3.3).

3.2 Adjusted ADM formulations

We show an application of the above idea, to the standard ADM system, Box 2.1.

3.2.1 Adjustments to ADM equations and its effects on constraint propagations

Generally, we can write the adjustment terms to (2.5) and (2.6) using (2.7) and (2.8) by the following combinations (using up to the first derivatives of constraints for simplicity):

The adjusted ADM formulation [78]: **Box 3.4**

Modify the evolution eqs of (γ_{ij}, K_{ij}) by constraints $\mathcal{H}$ and $\mathcal{M}_i$,

$$\text{adjustment terms of } \partial_t\gamma_{ij}: \quad +P_{ij}\mathcal{H} + Q^k{}_{ij}\mathcal{M}_k + p^k{}_{ij}(\nabla_k\mathcal{H}) + q^{kl}{}_{ij}(\nabla_k\mathcal{M}_l), \tag{3.7}$$

$$\text{adjustment terms of } \partial_t K_{ij}: \quad +R_{ij}\mathcal{H} + S^k{}_{ij}\mathcal{M}_k + r^k{}_{ij}(\nabla_k\mathcal{H}) + s^{kl}{}_{ij}(\nabla_k\mathcal{M}_l), \tag{3.8}$$

where P, Q, R, S and p, q, r, s are multipliers. That is,

$$\partial_t\gamma_{ij} = (2.5) + P_{ij}\mathcal{H} + Q^k{}_{ij}\mathcal{M}_k + p^k{}_{ij}(\nabla_k\mathcal{H}) + q^{kl}{}_{ij}(\nabla_k\mathcal{M}_l), \tag{3.9}$$

$$\partial_t K_{ij} = (2.6) + R_{ij}\mathcal{H} + S^k{}_{ij}\mathcal{M}_k + r^k{}_{ij}(\nabla_k\mathcal{H}) + s^{kl}{}_{ij}(\nabla_k\mathcal{M}_l). \tag{3.10}$$

According to this adjustment, the constraint propagation equations are also modified as

$$\partial_t\mathcal{H} = (2.9) + H_1^{mn}(3.7) + H_2^{imn}\partial_i(3.7) + H_3^{ijmn}\partial_i\partial_j(3.7) + H_4^{mn}(3.8), \tag{3.11}$$

$$\partial_t\mathcal{M}_i = (2.10) + M_{1i}{}^{mn}(3.7) + M_{2i}{}^{jmn}\partial_j(3.7) + M_{3i}{}^{mn}(3.8) + M_{4i}{}^{jmn}\partial_j(3.8). \tag{3.12}$$

with appropriate changes in indices. (See Appendix A. $H_1, \cdots, M_1, \cdots$ are defined there.)

We show two examples of adjustments here. We will list several others later in Table 2.

1. **The standard ADM vs. original ADM**
   The first comparison is to show the differences between the standard ADM [98] and the original ADM system [12] (see §2.0). In the notation of (3.7) and (3.8), the adjustment,

$$R_{ij} = \kappa_F \alpha \gamma_{ij}, \qquad \text{(and set the other multipliers zero)} \tag{3.13}$$

   will distinguish two, where κ_F is a constant. Here $\kappa_F = 0$ corresponds to the standard ADM (no adjustment), and $\kappa_F = -1/4$ to the original ADM (without any adjustment to the canonical formulation by ADM). As one can check by (3.11) and (3.12) adding R_{ij} term keeps the constraint propagation in a first-order form. Frittelli [39] (see also [95]) pointed out that the hyperbolicity of constraint propagation equations is better in the standard ADM system.

2. **Detweiler type**
   Detweiler [33] found that with a particular combination, the evolution of the energy norm of the constraints, $\mathcal{H}^2 + \mathcal{M}^2$, can be negative definite when we apply the maximal slicing condition, $K = 0$. His adjustment can be written in our notation in (3.7) and (3.8), as

$$\begin{aligned} P_{ij} &= -\kappa_L \alpha^3 \gamma_{ij}, & (3.14)\\ R_{ij} &= \kappa_L \alpha^3 (K_{ij} - (1/3) K \gamma_{ij}), & (3.15)\\ S^k{}_{ij} &= \kappa_L \alpha^2 [3(\partial_{(i}\alpha)\delta^k_{j)} - (\partial_l \alpha)\gamma_{ij}\gamma^{kl}], & (3.16)\\ s^{kl}{}_{ij} &= \kappa_L \alpha^3 [\delta^k_{(i}\delta^l_{j)} - (1/3)\gamma_{ij}\gamma^{kl}], & (3.17) \end{aligned}$$

   everything else is zero, where κ_L is a constant. Detweiler's adjustment, (3.14)-(3.17), does not put constraint propagation equation to a first order form, so we cannot discuss hyperbolicity or the characteristic speed of the constraints. We confirmed numerically, using perturbation on Minkowskii and Schwarzschild spacetime, that Detweiler's system provides better accuracy than the standard ADM, but only for small positive κ_L.

3.2.2 Procedure to evaluate constraint amplification factors in spherically symmetric spacetime

Before we compare between particular adjustment examples, we describe our procedure to evaluate CAFs in spherically symmetric spacetime. According to our motivation, the actual procedure to analyze the adjustments is to substitute the perturbed metric to the (adjusted) evolution equations first and to evaluate the according perturbative errors in the (adjusted) constraint propagation equations. However, for the simplicity, we apply the perturbation to the pair of constraints directly and analyze the effects of adjustments in its propagation equations. The latter, we think, presents the feature of constraint propagation more clearly for our purposes.

The discussion becomes clear if we expand the constraint $C_\mu := (\mathcal{H}, \mathcal{M}_i)^T$ using vector harmonics,

$$C_\mu = \sum_{l,m} \left(A^{lm} a_{lm} + B^{lm} b_{lm} + C^{lm} c_{lm} + D^{lm} d_{lm} \right), \tag{3.18}$$

where we choose the basis as

$$\begin{aligned} a_{lm}(\theta,\varphi) &= (Y_{lm}, 0, 0, 0)^T, & (3.19)\\ b_{lm}(\theta,\varphi) &= (0, Y_{lm}, 0, 0)^T, & (3.20)\\ c_{lm}(\theta,\varphi) &= \frac{r}{\sqrt{l(l+1)}}(0, 0, \partial_\theta Y_{lm}, \partial_\varphi Y_{lm})^T, & (3.21)\\ d_{lm}(\theta,\varphi) &= \frac{r}{\sqrt{l(l+1)}}(0, 0, -\frac{1}{\sin\theta}\partial_\varphi Y_{lm}, \sin\theta\, \partial_\theta Y_{lm})^T, & (3.22) \end{aligned}$$

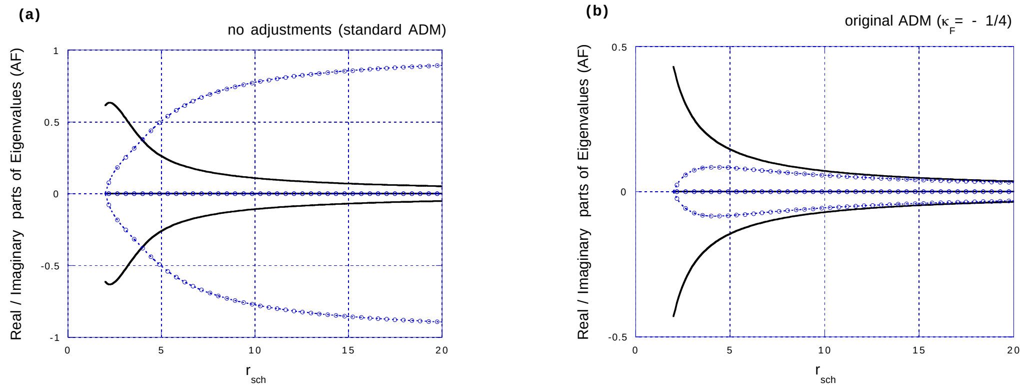

Figure 5: Amplification factors (CAFs, eigenvalues of homogenized constraint propagation equations) are shown for the standard Schwarzschild coordinate, with (a) no adjustments, i.e., standard ADM, and (b) original ADM ($\kappa_F = -1/4$). [see eq. (3.13)]. The solid lines and the dotted lines with circles are real parts and imaginary parts, respectively. They are four lines each, but actually the two eigenvalues are zero for all cases. Plotting range is $2 < r \leq 20$ using Schwarzschild radial coordinate. We set $k = 1, l = 2$, and $m = 2$ throughout the article. (Reprinted from [78], ©APS 2002)

and the coefficients $A^{lm}, \cdots, D^{lm}$ are functions of (t, r). Here Y_{lm} is the spherical harmonic function,

$$Y_{lm}(\theta, \varphi) = (-1)^{(m+|m|)/2} \sqrt{\frac{(2l+1)}{4\pi} \frac{(l-|m|)!}{(l+|m|)!}} P_l^m(\cos\theta) e^{im\varphi}. \tag{3.23}$$

The basis (3.19)-(3.22) are normalized so that they satisfy

$$\langle C_\mu, C_\nu \rangle = \int_0^{2\pi} d\varphi \int_0^{\pi} C_\mu^* C_\rho \, \eta^{\mu\rho} \sin\theta d\theta, \tag{3.24}$$

where $\eta^{\mu\rho}$ is Minkowskii metric and the asterisk denotes the complex conjugate. Therefore

$$A^{lm} = \langle a^{lm}_{(\nu)}, C_\nu \rangle, \quad \partial_t A^{lm} = \langle a^{lm}_{(\nu)}, \partial_t C_\nu \rangle, \quad \text{etc.} \tag{3.25}$$

In order to analyze the radial dependences, we also express these evolution equations using the Fourier expansion on the radial coordinate,

$$A^{lm} = \sum_k \hat{A}^{lm}_{(k)}(t) \, e^{ikr} \quad \text{etc.} \tag{3.26}$$

So that we can obtain the RHS of the evolution equations for $(\hat{A}^{lm}_{(k)}(t), \cdots, \hat{D}^{lm}_{(k)}(t))^T$ in a homogeneous form.

3.2.3 Constraint amplification factors in Schwarzschild spacetime

We present our CAF analysis in Schwarzschild black hole spacetime, which metric is

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{dr^2}{1 - 2M/r} + r^2 d\Omega^2, \qquad \text{(the standard expression)} \tag{3.27}$$

where M is the mass of a black hole. For numerical relativists, evolving a single black hole is the essential test problem, though it is a trivial at first sight. The standard expression, (3.27), has a

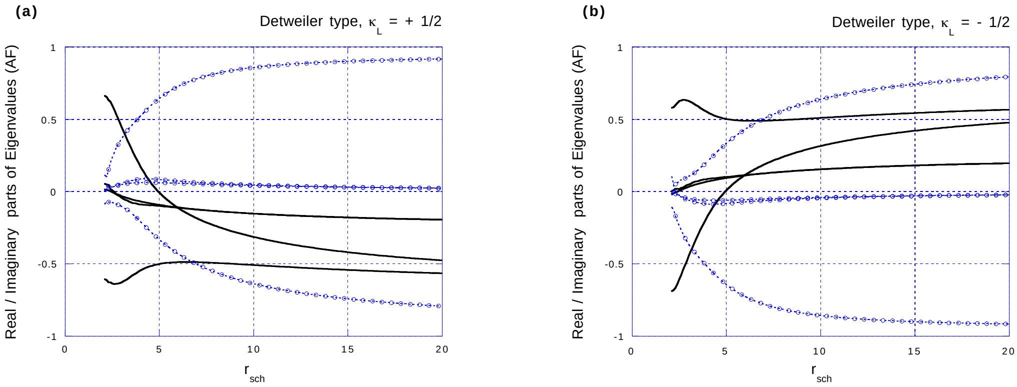

Figure 6: Amplification factors of the standard Schwarzschild coordinate, with Detweiler type adjustments, (3.14)-(3.17). Multipliers used in the plot are (a) $\kappa_L = +1/2$, and (b) $\kappa_L = -1/2$. Plotting details are the same as Fig.5. (Reprinted from [78], ©APS 2002)

coordinate singularity at $r = 2M$, so that we need to move another coordinate for actual numerical time integrations.

The ingoing Eddington-Finkelstein (iEF) coordinate has become popular in numerical relativity, in order to excise black hole singularity, since iEF penetrates the horizon without an irregular coordinate. The expression is,

$$ds^2 = -(1 - \frac{2M}{r})dt_{iEF}^2 + \frac{4M}{r}dt_{iEF}dr + (1 + \frac{2M}{r})dr^2 + r^2 d\Omega^2 \qquad \text{(the iEF expression)} \tag{3.28}$$

which is given by $t_{iEF} = t + 2M\log(r - 2M)$ and the radial coordinate is common to (3.27).

We show CAFs in the following cases. More examples are available in [78].

1. **in the standard Schwarzschild metric expression (3.27)**

   - **For the adjustment (3.13)**, CAFs are obtained as

     $$\begin{aligned} \Lambda^i &= (0, 0, \sqrt{a}, -\sqrt{a}), & (3.29)\\ a &= -k^2 + \frac{4Mk^2r^2(r-M) + 2M(2r-M) + l(l+1)r(r-2M) + ikr(2r^2 - 3Mr - 2M^2)}{r^4} \end{aligned}$$

     for the choice of $\kappa_1 = 0$ (the standard ADM), while they are

     $$\Lambda^i = (0, 0, \sqrt{b}, -\sqrt{b}), \qquad b = \frac{M(2r-M) + irkM(2M-r)}{r^4} \tag{3.30}$$

     for the choice of $\kappa_1 = -1/4$ (the original ADM).

     These are plotted in Fig.5. The solid lines and dotted lines with circles are real parts and imaginary parts of CAFs, respectively. They are four lines each, but as we showed in (3.29), two of them are zero. The plotting range is $2 < r \le 20$ in Schwarzschild radial coordinate. The CAFs at $r = 2$ are $\pm\sqrt{3/8}$ and 0. The existence of this positive real CAF near the horizon is an important result. We show only the cases with $l = 2$ and $k = 1$, because we judged that the plots of $l = 0$ and other ks are qualitatively the same.

     The adjustment (3.13) with $\kappa_F = -1/4$ returns the system back to the original ADM. CAFs are (3.30) and we plot them in Fig.5(b). We can see that the imaginary parts are apparently different from those of the standard ADM [Fig.5(a)]. This is the same feature as

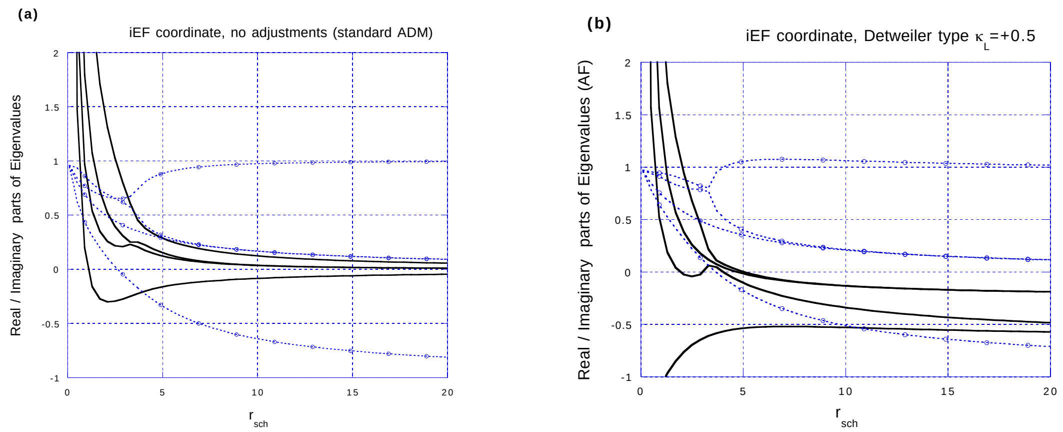

Figure 7: CAFs in the iEF coordinate (3.28) on $t = 0$ slice for the standard ADM formulation (i.e. no adjustments) [See Fig.5(a) for the standard Schwarzschild coordinate.] and for Detweiler adjustments with $\kappa_L = +1/2$ [See Fig.6(a) for the standard Schwarzschild coordinate.]. The solid four lines and the dotted four lines with circles are real parts and imaginary parts, respectively. (Reprinted from [78], ©APS 2002)

in the case of the flat background [95]. According to our conjecture, the non-zero imaginary values are better than zeros, so we expect that the standard ADM has a better evolution property than the original ADM system. Negative κ_F always makes the asymptotical real values finite.

- **The Detweiler-type adjustment (3.14)-(3.17)** makes the feature completely different. Fig.6(a) and (b) are the cases of $\kappa_L = \pm 1/2$. A great improvement can be seen in the positive κ_L case where *all* real parts become negative in large r. Moreover all imaginary parts are apart from zero. These are the desired features according to our conjecture. Therefore we expect the Detweiler adjustment has good stability properties *except* near the black hole region. The CAF near the horizon *has* a positive real component. This is not contradictory with the Detweiler's original idea. His idea came from suppressing the *total* L2 norm of constraints on the spatial slice, while our plot indicates the existence of a *local* violation mode. The change of signature of κ_L can be understood just by changing the signature of CAFs, and this fact can also be seen to the other plot. In [78] we reported that a partial adjustment (apply only (3.14) or (3.17)) is also effective.

2. **in the iEF coordinates (3.28)**

   - **For the adjustment (3.13)**, we plotted CAFs in Figure 7. We Figure 7(a) is qualitatively different from Fig.5(a). This is because the iEF expression is asymmetric to time, i.e. has non-zero extrinsic curvature. We notice that while some CAFs in iEF remain positive in large r region, that their nature changes due to the adjustments.
   - **For the Detweiler-type adjustment (3.14)-(3.17)**, CAFs are as in Figure 7(b). Interestingly, all plots indicate that all real parts of CAFs are negative, and imaginary parts are non-zero (again except near the black hole region). By arranging the multiplier parameter, there is a chance to get all negative real CAFs outside the black hole horizon. For example, all the real-part goes negative outside the black hole horizon if $\kappa_L > 3.1$, while large κ_L may introduce another instability problem [94].

Such kinds of test can be done with other combinations. In Table 2, we listed our results for more examples. We defined the adjustment terms so that their positive multiplier parameter, $\kappa > 0$, makes

No.	adjustment		1st?	Sch coord.			iEF coord.	
				TRS	real.	imag.	real.	imag.
0	–	no adjustments	yes	–	–	–	–	–
P-1	P_{ij}	$-\kappa_L\alpha^3\gamma_{ij}$	no	no	makes 2 Neg.	not app.	makes 2 Neg.	not app.
P-2	P_{ij}	$-\kappa_L\alpha\gamma_{ij}$	no	no	makes 2 Neg.	not app.	makes 2 Neg.	not app.
P-3	P_{ij}	$P_{rr}=-\kappa$	no	no	slightly enl.Neg.	not app.	slightly enl.Neg.	not app.
P-4	P_{ij}	$-\kappa\gamma_{ij}$	no	no	makes 2 Neg.	not app.	makes 2 Neg.	not app.
P-5	P_{ij}	$-\kappa\gamma_{rr}$	no	no	red. Pos./enl.Neg.	not app.	red.Pos./enl.Neg.	not app.
Q-1	$Q^k{}_{ij}$	$\kappa\alpha\beta^k\gamma_{ij}$	no	no	N/A	N/A	$\kappa\sim 1.35$ min.vals.	not app.
Q-2	$Q^k{}_{ij}$	$Q^r{}_{rr}=\kappa$	no	yes	red. abs vals.	not app.	red. abs vals.	not app.
Q-3	$Q^k{}_{ij}$	$Q^r{}_{ij}=\kappa\gamma_{ij}$	no	yes	red. abs vals.	not app.	enl.Neg.	enl. vals.
Q-4	$Q^k{}_{ij}$	$Q^r{}_{rr}=\kappa\gamma_{rr}$	no	yes	red. abs vals.	not app.	red. abs vals.	not app.
R-1	R_{ij}	$\kappa_F\alpha\gamma_{ij}$	yes	yes	$\kappa_F=-1/4$ min. abs vals.		$\kappa_F=-1/4$ min. vals.	
R-2	R_{ij}	$R_{rr}=-\kappa_\mu\alpha$	yes	no	not app.	not app.	red.Pos./enl.Neg.	enl. vals.
R-3	R_{ij}	$R_{rr}=-\kappa\gamma_{rr}$	yes	no	enl. vals.	not app.	red.Pos./enl.Neg.	enl. vals.
S-1	$S^k{}_{ij}$	$S^k{}_{ij}=(3.16)$	yes	no	not app.	not app.	not app.	not app.
S-2	$S^k{}_{ij}$	$\kappa\alpha\gamma^{lk}(\partial_l\gamma_{ij})$	yes	no	makes 2 Neg.	not app.	makes 2 Neg.	not app.
p-1	$p^k{}_{ij}$	$p^r{}_{ij}=-\kappa\alpha\gamma_{ij}$	no	no	red. Pos.	red. vals.	red. Pos.	enl. vals.
p-2	$p^k{}_{ij}$	$p^r{}_{rr}=\kappa\alpha$	no	no	red. Pos.	red. vals.	red.Pos/enl.Neg.	enl. vals.
p-3	$p^k{}_{ij}$	$p^r{}_{rr}=\kappa\alpha\gamma_{rr}$	no	no	makes 2 Neg.	enl.vals.	red.Pos.vals.	red.vals.
q-1	$q^{kl}{}_{ij}$	$q^{rr}{}_{ij}=\kappa\alpha\gamma_{ij}$	no	no	$\kappa=1/2$ min.vals.	red.vals.	not app.	enl.vals.
q-2	$q^{kl}{}_{ij}$	$q^{rr}{}_{rr}=-\kappa\alpha\gamma_{rr}$	no	yes	red. abs vals.	not app.	not app.	not app.
r-1	$r^k{}_{ij}$	$r^r{}_{ij}=\kappa\alpha\gamma_{ij}$	no	yes	not app.	not app.	not app.	enl. vals.
r-2	$r^k{}_{ij}$	$r^r{}_{rr}=-\kappa\alpha$	no	yes	red. abs vals.	enl. vals.	red. abs vals.	enl. vals.
r-3	$r^k{}_{ij}$	$r^r{}_{rr}=-\kappa\alpha\gamma_{rr}$	no	yes	red. abs vals.	enl. vals.	red. abs vals.	enl. vals.
s-1	$s^{kl}{}_{ij}$	$s^{kl}{}_{ij}=(3.17)$	no	no	makes 4 Neg.	not app.	makes 4 Neg.	not app.
s-2	$s^{kl}{}_{ij}$	$s^{rr}{}_{ij}=-\kappa\alpha\gamma_{ij}$	no	no	makes 2 Neg.	red. vals.	makes 2 Neg.	red. vals.
s-3	$s^{kl}{}_{ij}$	$s^{rr}{}_{rr}=-\kappa\alpha\gamma_{rr}$	no	no	makes 2 Neg.	red. vals.	makes 2 Neg.	red. vals.

Table 2: List of ADM adjustments we tested in the Schwarzschild spacetime. The column of adjustments are nonzero multipliers in terms of (3.7) and (3.8). The column 'TRS' indicates whether each adjusting term satisfies the time reversal symmetry or not on the standard Schwarzschild coordinate. ('No' means a candidate that makes asymmetric CAFs.) The column '1st?' indicates whether each adjusting term breaks the first-order feature of the standard constraint propagation equation, (2.9) and (2.10). ('Yes' keeps the system first-order, 'No' may break hyperbolicity of constraint propagation depending a choice of κ.) The effects to CAFs (when $\kappa>0$) are commented for both coordinates and both real/imaginary parts, respectively. The 'N/A' means that there is no effect due to the coordinate properties; 'not app.' (not apparent) means the adjustment does not change the CAFs effectively according to our conjecture; 'enl./red./min.' means enlarge/reduce/minimize, and 'Pos./Neg.' means positive/negative, respectively. These judgements are made at the $r\sim O(10M)$ region on their $t=0$ slice. See more detail in [78].

the system *better* in stability according to our conjecture. (Here *better* means in accordance with our conjecture in Box 3.2 and 3.3). The table includes the above results and is intended to extract the contributions of each term in (3.7) and (3.8). The effects of adjustments (of each $\kappa>0$ case) to CAFs are commented upon for each coordinate system and for real/imaginary parts of CAFs, respectively. These judgements are made at the $r\sim O(10M)$ region on their $t=0$ slice. Among them, No. R-2 in Table 2 explains why a particular adjustment by PennState group [51] gives better stability than before.

3.2.4 Remarks

Numerical demonstrations The above analyses are only predictions, and supporting numerical demonstrations are necessary for the next steps. Systematic numerical comparisons are progressing, and we show two sample plots here.

Figure 8 (a) is a test numerical evolution of Detweiler-type adjustment on the Minkowskii background. We see the adjusted version gives convergence on to the constraint surface by arranging the

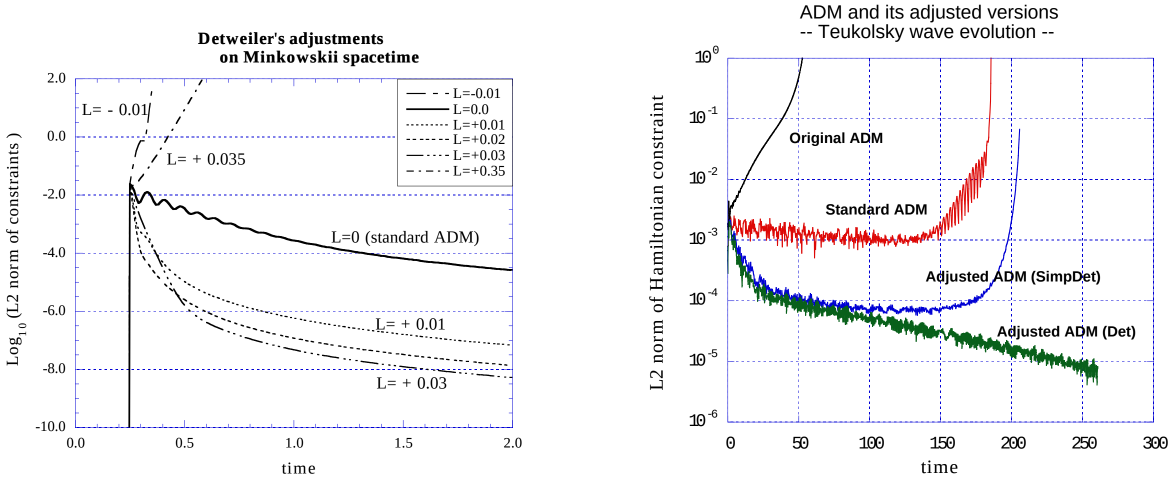

Figure 8: Comparisons of numerical evolution between adjusted ADM systems. (a) Demonstration of the Detweiler's modified ADM system on Minkowskii background spacetime, 1-dimensional simulation. The L2 norm of the constraints $\mathcal{H}^{ADM}$ and $\mathcal{M}^{ADM}$ is plotted in the function of time. Artificial error was added at $t = 0.25$. L is the parameter used in (3.14)-(3.16). We see the evolution is asymptotically constrained for small $\kappa > 0$. (Reprinted from [95], ©APS 2001) (b) L2 norm of the Hamiltonian constraint $\mathcal{H}^{ADM}$ of evolution using ADM/adjusted ADM formulations for the case of Teukolsky wave, 3-dimensional simulation. Cactus-based original ADM module (CactusGR) was used. (Shinkai-Yoneda, in preparation).

magnitude of the adjusting parameter, κ.

Figure 8 (b) is an example from the project of "Comparison of formulations of the Einstein equations for numerical relativity" [59]. (We will describe the project in §4). The plot was obtained by a 3-dimensional numerical evolution of weak gravitational wave, the so-called Teukolsky wave [85]. The lines are of the original/standard ADM evolution equations, Detweiler-type adjustment, and a part of Detweiler-type adjustment (actually used only (3.14)). For a particular choice of κ, we observe again the L2 norm of constraint (violation of constraints) is reduced than the standard ADM case, and can evolve longer than that.

Notion of Time Reversal Symmetry During the comparisons of adjustments, we found that it is necessary to create time asymmetric structure of evolution equations in order to force the evolution on to the constraint surface. There are infinite ways of adjusting equations, but we found that if we follow the guideline Box 3.5, then such an adjustment will give us time asymmetric evolution.

Trick to obtain asymptotically constrained system: **Box 3.5**
= Break the time reversal symmetry (TRS) of the evolution equation.

1. Evaluate the parity of the evolution equation.
   By reversing the time ($\partial_t \to -\partial_t$), there are variables which change their signatures (parity $(-)$) [e.g. $K_{ij}, \partial_t \gamma_{ij}, \mathcal{M}_i, \cdots$], while not (parity $(+)$) [e.g. $g_{ij}, \partial_t K_{ij}, \mathcal{H}, \cdots$].

2. Add adjustments which have different parity of that equation.
   For example, for the parity $(-)$ equation $\partial_t \gamma_{ij}$, add a parity $(+)$ adjustment $\kappa \mathcal{H}$.

One of our criteria, the negative real CAFs, requires breaking the time-symmetric features of the original evolution equations. Such CAFs are obtained by adjusting the terms which break the TRS of the evolution equations, and this is available even at the standard ADM system. The TRS features (on the standard Schwarzschild metric, which is time-symmetric background) are also denoted in Table 2. The criteria of Box 3.5 will be applied also to the BSSN equation (see §3.3.3).

Differences with Detweiler's requirement We comment on the differences between Detweiler's criteria [33] and ours. Detweiler calculated the L2 norm of the constraints, C_ρ, over the 3-hypersurface and imposed the negative definiteness of its evolution,

$$\text{Detweiler's criteria} \;\Leftrightarrow\; \partial_t \int C_\rho C^\rho \; dV < 0, \;\; \forall \text{ non zero } C_\rho. \tag{3.31}$$

where $C_\rho C^\rho =: G^{\rho\sigma} C_\rho C_\sigma$, and $G_{\rho\sigma} = diag[1, \gamma_{ij}]$ for the pair of $C_\rho = (\mathcal{H}, \mathcal{M}_i)$.

Assuming the constraint propagation to be $\partial_t \hat{C}_\rho = A_\rho{}^\sigma \hat{C}_\sigma$ in the Fourier components, the time derivative of the L2 norm can be written as

$$\partial_t(\hat{C}_\rho \hat{C}^\rho) = (A^{\rho\sigma} + \bar{A}^{\sigma\rho} + \partial_t \bar{G}^{\rho\sigma})\hat{C}_\rho \bar{\hat{C}}_\sigma. \tag{3.32}$$

Together with the fact that the L2 norm is preserved by Fourier transform, we can say, for the case of *static* background metric,

$$\begin{aligned} \text{Detweiler's criteria} \quad &\Leftrightarrow \quad \text{eigenvalues of } (A + A^\dagger) \text{ are all negative } \forall k. &(3.33)\\ \text{Our criteria} \quad &\Leftrightarrow \quad \text{eigenvalues of } A \text{ are all negative } \forall k. &(3.34) \end{aligned}$$

Therefore for the case of static background, Detweiler's criterion is stronger than ours. For example, the matrix

$$A = \begin{pmatrix} -1 & a \\ 0 & -1 \end{pmatrix} \quad \text{where } a \text{ is constant,} \tag{3.35}$$

for the evolution system $(\hat{C}_1, \hat{C}_2)$ satisfies our criterion but not Detweiler's when $|a| \geq \sqrt{2}$. This matrix however gives asymptotical decay for $(\hat{C}_1, \hat{C}_2)$. Therefore we may say that Detweiler requires the monotonic decay of the constraints, while we assume only asymptotical decay.

We remark that Detweiler's truncations on higher order terms in C-norm corresponds to our perturbational analysis; both are based on the idea that the deviations from constraint surface (the errors expressed non-zero constraint value) are initially small.

Ranges of effective κ We do not discuss the ranges of the effective multiplier parameter, κ, since the range depends on the characteristic speeds of the models and numerical integration schemes as we observed in [94]. At this moment, we can only estimate the maximum value for adjusting parameter κs from von Neumann's stability analysis, while we do not have a background theoretical explanation to predict the optimized value of κ for numerical evolution. We will comment this point later again in §4.

3.3 Adjusted BSSN formulations

Next we apply the idea of adjusted system to the BSSN system, Box 2.3.

3.3.1 Constraint propagation analysis of the BSSN equations

The BSSN has 5 constraint equations, $(\mathcal{H}, \mathcal{M}_i, \mathcal{G}^i, \mathcal{A}, \mathcal{S})$, see (2.26)-(2.30). We begin identifying where the BSSN evolution equations were adjusted already in its standard notation, (2.21)-(2.25).

Taking careful account of these constraints, (2.26) and (2.27) can be expressed directly as

$$\mathcal{H} = e^{-4\varphi}\tilde{R} - 8e^{-4\varphi}\tilde{D}^j\tilde{D}_j\varphi - 8e^{-4\varphi}(\tilde{D}^j\varphi)(\tilde{D}_j\varphi) + (2/3)K^2 - \tilde{A}_{ij}\tilde{A}^{ij} - (2/3)\mathcal{A}K, \quad (3.36)$$

$$\mathcal{M}_i = 6\tilde{A}^j{}_i(\tilde{D}_j\varphi) - 2\mathcal{A}(\tilde{D}_i\varphi) - (2/3)(\tilde{D}_iK) + \tilde{\gamma}^{kj}(\tilde{D}_j\tilde{A}_{ki}). \quad (3.37)$$

By a straightforward calculation, we get:

$$\begin{aligned}
\partial_t^B\varphi &= \partial_t^A\varphi + (1/6)\alpha\mathcal{A} - (1/12)\tilde{\gamma}^{-1}(\partial_j\mathcal{S})\beta^j, & (3.38)\\
\partial_t^B\tilde{\gamma}_{ij} &= \partial_t^A\tilde{\gamma}_{ij} - (2/3)\alpha\tilde{\gamma}_{ij}\mathcal{A} + (1/3)\tilde{\gamma}^{-1}(\partial_k\mathcal{S})\beta^k\tilde{\gamma}_{ij}, & (3.39)\\
\partial_t^B K &= \partial_t^A K - (2/3)\alpha K\mathcal{A} - \alpha\mathcal{H} + \alpha e^{-4\varphi}(\tilde{D}_j\mathcal{G}^j), & (3.40)\\
\partial_t^B\tilde{A}_{ij} &= \partial_t^A\tilde{A}_{ij} + ((1/3)\alpha\tilde{\gamma}_{ij}K - (2/3)\alpha\tilde{A}_{ij})\mathcal{A} & \\
&\quad +((1/2)\alpha e^{-4\varphi}(\partial_k\tilde{\gamma}_{ij}) - (1/6)\alpha e^{-4\varphi}\tilde{\gamma}_{ij}\tilde{\gamma}^{-1}(\partial_k\mathcal{S}))\mathcal{G}^k & \\
&\quad +\alpha e^{-4\varphi}\tilde{\gamma}_{k(i}(\partial_{j)}\mathcal{G}^k) - (1/3)\alpha e^{-4\varphi}\tilde{\gamma}_{ij}(\partial_k\mathcal{G}^k), & (3.41)\\
\partial_t^B\tilde{\Gamma}^i &= \partial_t^A\tilde{\Gamma}^i + (\,-(2/3)(\partial_j\alpha)\tilde{\gamma}^{ji} - (2/3)\alpha(\partial_j\tilde{\gamma}^{ji}) - (1/3)\alpha\tilde{\gamma}^{ji}\tilde{\gamma}^{-1}(\partial_j\mathcal{S}) + 4\alpha\tilde{\gamma}^{ij}(\partial_j\varphi))\mathcal{A} & \\
&\quad -(2/3)\alpha\tilde{\gamma}^{ji}(\partial_j\mathcal{A}) + 2\alpha\tilde{\gamma}^{ij}\mathcal{M}_j - (1/2)(\partial_k\beta^i)\tilde{\gamma}^{kj}\tilde{\gamma}^{-1}(\partial_j\mathcal{S}) + (1/6)(\partial_j\beta^k)\tilde{\gamma}^{ij}\tilde{\gamma}^{-1}(\partial_k\mathcal{S}) & \\
&\quad +(1/3)(\partial_k\beta^k)\tilde{\gamma}^{ij}\tilde{\gamma}^{-1}(\partial_j\mathcal{S}) + (5/6)\beta^k\tilde{\gamma}^{-2}\tilde{\gamma}^{ij}(\partial_k\mathcal{S})(\partial_j\mathcal{S}) + (1/2)\beta^k\tilde{\gamma}^{-1}(\partial_k\tilde{\gamma}^{ij})(\partial_j\mathcal{S}) & \\
&\quad +(1/3)\beta^k\tilde{\gamma}^{-1}(\partial_j\tilde{\gamma}^{ji})(\partial_k\mathcal{S}). & (3.42)
\end{aligned}$$

where ∂_t^A denotes the part of no replacements, i.e. the terms only use the standard ADM evolution equations in its time derivatives.

From (3.38)-(3.42), we understand that all BSSN evolution equations are *adjusted* using constraints. However, from the viewpoint of time reversal symmetry (Box 3.5), all the above adjustments in (3.38)-(3.42) unfortunately keep the time reversal symmetry. Therefore we can not expect direct decays of constraint violation in the present form.

The set of the constraint propagation equations, $\partial_t(\mathcal{H}, \mathcal{M}_i, \mathcal{G}^i, \mathcal{A}, \mathcal{S})^T$, turns to be not a first-order hyperbolic and includes many non-linear terms, (see the Appendix in [96]). In order to understand the fundamental structure, we show an analysis on the flat spacetime background.

For the flat background metric $g_{\mu\nu} = \eta_{\mu\nu}$, the first order perturbation equations of (3.38)-(3.42) can be written as

$$\begin{aligned}
\partial_t{}^{(1)}\!\varphi &= -(1/6)^{(1)}\!K + (1/6)(\kappa_\varphi - 1)^{(1)}\!\mathcal{A}, & (3.43)\\
\partial_t{}^{(1)}\!\tilde{\gamma}_{ij} &= -2^{(1)}\!\tilde{A}_{ij} - (2/3)(\kappa_{\tilde{\gamma}} - 1)\delta_{ij}{}^{(1)}\!\mathcal{A}, & (3.44)\\
\partial_t{}^{(1)}\!K &= -(\partial_j\partial_j{}^{(1)}\!\alpha) + (\kappa_{K1} - 1)\partial_j{}^{(1)}\!\mathcal{G}^j - (\kappa_{K2} - 1)^{(1)}\!\mathcal{H}, & (3.45)\\
\partial_t{}^{(1)}\!\tilde{A}_{ij} &= {}^{(1)}\!(R_{ij}^{BSSN})^{TF} - {}^{(1)}\!(\tilde{D}_i\tilde{D}_j\alpha)^{TF} + (\kappa_{A1} - 1)\delta_{k(i}(\partial_{j)}{}^{(1)}\!\mathcal{G}^k) - (1/3)(\kappa_{A2} - 1)\delta_{ij}(\partial_k{}^{(1)}\!\mathcal{G}^k), & (3.46)\\
\partial_t{}^{(1)}\!\tilde{\Gamma}^i &= -(4/3)(\partial_i{}^{(1)}\!K) - (2/3)(\kappa_{\tilde{\Gamma}1} - 1)(\partial_i{}^{(1)}\!\mathcal{A}) + 2(\kappa_{\tilde{\Gamma}2} - 1)^{(1)}\!\mathcal{M}_i, & (3.47)
\end{aligned}$$

where we introduced parameters κs, all $\kappa = 0$ reproduce no adjustment case from the standard ADM equations, and all $\kappa = 1$ correspond to the BSSN equations. We express them as

$$\kappa_{adj} := (\kappa_\varphi, \kappa_{\tilde{\gamma}}, \kappa_{K1}, \kappa_{K2}, \kappa_{A1}, \kappa_{A2}, \kappa_{\tilde{\Gamma}1}, \kappa_{\tilde{\Gamma}2}). \quad (3.48)$$

Constraint propagation equations at the first order in the flat spacetime, then, become:

$$\begin{aligned}
\partial_t{}^{(1)}\!\mathcal{H} &= (\kappa_{\tilde{\gamma}} - (2/3)\kappa_{\tilde{\Gamma}1} - (4/3)\kappa_\varphi + 2)\,\partial_j\partial_j{}^{(1)}\!\mathcal{A} + 2(\kappa_{\tilde{\Gamma}2} - 1)(\partial_j{}^{(1)}\!\mathcal{M}_j), & (3.49)\\
\partial_t{}^{(1)}\!\mathcal{M}_i &= (-(2/3)\kappa_{K1} + (1/2)\kappa_{A1} - (1/3)\kappa_{A2} + (1/2))\,\partial_i\partial_j{}^{(1)}\!\mathcal{G}^j & \\
&\quad +(1/2)\kappa_{A1}\partial_j\partial_j{}^{(1)}\!\mathcal{G}^i + ((2/3)\kappa_{K2} - (1/2))\,\partial_i{}^{(1)}\!\mathcal{H}, & (3.50)\\
\partial_t{}^{(1)}\!\mathcal{G}^i &= 2\kappa_{\tilde{\Gamma}2}{}^{(1)}\!\mathcal{M}_i + (-(2/3)\kappa_{\tilde{\Gamma}1} - (1/3)\kappa_{\tilde{\gamma}})(\partial_i{}^{(1)}\!\mathcal{A}), & (3.51)\\
\partial_t{}^{(1)}\!\mathcal{S} &= -2\kappa_{\tilde{\gamma}}{}^{(1)}\!\mathcal{A}, & (3.52)\\
\partial_t{}^{(1)}\!\mathcal{A} &= (\kappa_{A1} - \kappa_{A2})(\partial_j{}^{(1)}\!\mathcal{G}^j). & (3.53)
\end{aligned}$$

3.3.2 The origin of the advantages of the BSSN equations

We next discuss CAFs of (3.49)-(3.53). Hereafter we let $k^2 = k_x^2 + k_y^2 + k_z^2$ for Fourier wave numbers.

1. The no-adjustment case, κ_{adj} =(all zeros). This is the starting point of the discussion. In this case,
$$\text{CAFs} = (0\,(\times 7), \pm\sqrt{-k^2}),$$
i.e., $(0\,(\times 7), \pm\text{pure imaginary}\,(1\text{ pair}))$. In the standard ADM formulation, which uses (γ_{ij}, K_{ij}), CAFs are $(0, 0, \pm\text{Pure Imaginary})$ [95]. Therefore if we do not apply adjustments in BSSN equations the constraint propagation structure is quite similar to that of the standard ADM.

2. For the BSSN equations, κ_{adj} =(all 1s),
$$\text{CAFs} = (0\,(\times 3), \pm\sqrt{-k^2}\,(3\text{ pairs})),$$
i.e., $(0\,(\times 3), \pm\text{Pure Imaginary}\,(3\text{ pairs}))$. The number of pure imaginary CAFs is increased over that of No.1, and we conclude this is the advantage of adjustments used in BSSN equations.

3. No $\mathcal{S}$-adjustment case. All the numerical experiments so far apply the scaling condition $\mathcal{S}$ for the conformal factor φ. The $\mathcal{S}$-originated terms appear many places in BSSN equations (2.21)-(2.25), so that we guess non-zero $\mathcal{S}$ is a kind of source of the constraint violation. However, since all $\mathcal{S}$-originated terms do not appear in the flat spacetime background analysis, [no adjusted terms in (3.43)-(3.47)], our case can not say any effects due to S-constraint.

4. No $\mathcal{A}$-adjustment case. The trace (or traceout) condition for the variables is also considered necessary (e.g. [3]). This can be checked with $\kappa_{adj} = (\kappa, \kappa, 1, 1, 1, 1, \kappa, 1)$, and we get
$$\text{CAFs} = (0\,(\times 3), \pm\sqrt{-k^2}\,(3\text{ pairs})),$$
independent of κ. Therefore the effect of $\mathcal{A}$-adjustment is not apparent from this analysis.

5. No $\mathcal{G}^i$-adjustment case. The introduction of Γ^i is the key in the BSSN system. If we do not apply adjustments by $\mathcal{G}^i$, ($\kappa_{adj} = (1, 1, 0, 1, 0, 0, 1, 1)$) then we get
$$\text{CAFs} = (0(\times 7), \pm\sqrt{-k^2}),$$
which is the same with No.1. That is, adjustments due to $\mathcal{G}^i$ terms are effective to make a progress from ADM.

6. No $\mathcal{M}_i$-adjustment case. This can be checked with $\kappa_{adj} = (1, 1, 1, 1, 1, 1, 1, \kappa)$, and we get
$$\begin{aligned}\text{CAFs} &= (0, \pm\sqrt{-\kappa k^2}\,(2\text{ pairs}),\\ &\quad \pm\sqrt{-k^2(-1+4\kappa+|1-4\kappa|)/6}, \quad \pm\sqrt{-k^2(-1+4\kappa-|1-4\kappa|)/6}).\end{aligned}$$
If $\kappa = 0$, then $(0(\times 7), \pm\sqrt{k^2/3})$, which is $(0(\times 7), \pm\text{real value})$. Interestingly, these real values indicate the existence of the error growing mode together with the decaying mode. Alcubierre et al. [5] found that the adjustment due to the momentum constraint is crucial for obtaining stability. We think that they picked up this error growing mode. Fortunately at the BSSN limit ($\kappa = 1$), this error growing mode disappears and turns into a propagation mode.

7. No $\mathcal{H}$-adjustment case. The set $\kappa_{adj} = (1, 1, 1, \kappa, 1, 1, 1, 1)$ gives
$$\text{CAFs} = (0\,(\times 3), \pm\sqrt{-k^2}\,(3\text{ pairs})),$$
independently to κ. Therefore the effect of $\mathcal{H}$-adjustment is not apparent from this analysis.

No.		Constraints (number of components) $\mathcal{H}$ (1)	$\mathcal{M}_i$ (3)	$\mathcal{G}^i$ (3)	$\mathcal{A}$ (1)	$\mathcal{S}$ (1)	diag?	CAFs in Minkowskii background
0.	standard ADM	use	use	-	-	-	yes	$(0,0,\Im,\Im)$
1.	BSSN no adjustment	use	use	use	use	use	yes	$(0,0,0,0,0,0,\Im,\Im)$
2.	the BSSN	use+adj	use+adj	use+adj	use+adj	use+adj	no	$(0,0,0,\Im,\Im,\Im,\Im,\Im,\Im)$
3.	no $\mathcal{S}$ adjustment	use+adj	use+adj	use+adj	use+adj	use	no	no difference in flat background
4.	no $\mathcal{A}$ adjustment	use+adj	use+adj	use+adj	use	use+adj	no	$(0,0,0,\Im,\Im,\Im,\Im,\Im,\Im)$
5.	no $\mathcal{G}^i$ adjustment	use+adj	use+adj	use	use+adj	use+adj	no	$(0,0,0,0,0,0,0,\Im,\Im)$
6.	no $\mathcal{M}_i$ adjustment	use+adj	use	use+adj	use+adj	use+adj	no	$(0,0,0,0,0,0,0,\Re,\Re)$
7.	no $\mathcal{H}$ adjustment	use	use+adj	use+adj	use+adj	use+adj	no	$(0,0,0,\Im,\Im,\Im,\Im,\Im,\Im)$

Table 3: Contributions of adjustments terms and effects of introductions of new constraints in the BSSN system. The center column indicates whether each constraints are taken as a component of constraints in each constraint propagation analysis ('use'), and whether each adjustments are on ('adj'). The column 'diag?' indicates diagonalizability of the constraint propagation matrix. The right column shows amplification factors, where $\Im$ and $\Re$ means pure imaginary and real eigenvalue, respectively.

These tests are on the effects of adjustments that are already in the BSSN equations. We will consider whether much better adjustments are possible in the next section.

We list the above results in Table 3. The most characteristic points of the above are No. 5 and No.6 that denote the contribution of the momentum constraint adjustment and the importance of the new variable $\tilde{\Gamma}^i$. It is quite interesting that the unadjusted BSSN equations (case 2) does not have apparent advantages from the ADM system. As we showed in the case 5 and 6, if we missed a particular adjustment, then the expected stability behaviour occationally gets worse than the starting ADM system. Therefore we conclude that the better stability of the BSSN formulation is obtained by their adjustments in the equations, and the combination of the adjustments is in a good balance.

3.3.3 Proposals of the modified BSSN equations

We next consider the possibility whether we can obtain a system which has much better properties; whether more pure imaginary CAFs or negative real CAFs.

Heuristic examples

(A) A system which has 8 pure imaginary CAFs:

One direction is to seek a set of equations which make fewer zero CAFs than the standard BSSN case. Using the same set of adjustments in (3.43)-(3.47), CAFs are written as

$$\text{CAFs} = \Big(0, \pm\sqrt{-k^2\kappa_{A1}\kappa_{\tilde{\Gamma}2}}\,(2 \text{ pairs}), \pm\text{complicated expression}, \pm\text{complicated expression}\Big).$$

The terms in the first line certainly give four pure imaginary CAFs (two positive and negative real pairs) if $\kappa_{A1}\kappa_{\tilde{\Gamma}2} > 0\,(< 0)$. Keeping this in mind, by choosing $\kappa_{adj} = (1,1,1,1,1,\kappa,1,1)$, we find

$$\text{CAFs} = \Big(0, \pm\sqrt{-k^2}\,(2 \text{ pairs}), \pm\sqrt{-k^2(2+\kappa+|\kappa-4|)/6}, \pm\sqrt{-k^2(2+\kappa-|\kappa-4|)/6},\Big).$$

Therefore the adjustment $\kappa_{adj} = (1,1,1,1,1,4,1,1)$ gives CAFs $= \Big(0, \pm\sqrt{-k^2}\,(4 \text{ pairs})\Big)$, which is one step advanced from BSSN according our guidelines.

We note that such a system can be obtained in many ways, e.g. $\kappa_{adj} = (0,0,1,0,2,1,0,1/2)$ also gives four pairs of pure imaginary CAFs.

(B) A system which has negative real CAF:
One criterion to obtain a decaying constraint mode (i.e. an asymptotically constrained system) is to adjust an evolution equation as it breaks time reversal symmetry (Box 3.5). For example, we consider an additional adjustment to the BSSN equation as

$$\partial_t \tilde{\gamma}_{ij} = \partial_t^B \tilde{\gamma}_{ij} + \kappa_{SD} \alpha \tilde{\gamma}_{ij} \mathcal{H}, \tag{3.54}$$

which is a similar adjustment of the simplified Detweiler-type [33]. The constaint amplification factors become

$$\text{CAFs} = (0\,(\times 2), \pm\sqrt{-k^2}(3 \text{ pairs}), (3/2)k^2\kappa_{SD}),$$

in which the last one becomes negative real if $\kappa_{SD} < 0$.

(C) Combination of above (A) and (B)
Naturally we next consider both adjustments:

$$\begin{aligned} \partial_t \tilde{\gamma}_{ij} &= \partial_t^B \tilde{\gamma}_{ij} + \kappa_{SD} \alpha \tilde{\gamma}_{ij} \mathcal{H} && (3.55) \\ \partial_t \tilde{A}_{ij} &= \partial_t^B \tilde{A}_{ij} - \kappa_8 \alpha e^{-4\varphi} \tilde{\gamma}_{ij} \partial_k \mathcal{G}^k && (3.56) \end{aligned}$$

where the second one produces the 8 pure imaginary CAFs. We then obtain

$$\text{CAFs} = \left(0, \pm\sqrt{-k^2}\,(3 \text{ pairs}), (3/4)k^2\kappa_{SD} \pm \sqrt{k^2(-\kappa_8 + (9/16)k^2\kappa_{SD})}\right)$$

which reproduces case (A) when $\kappa_{SD} = 0, \kappa_8 = 1$, and case (B) when $\kappa_8 = 0$. These CAFs can become (0, pure imaginary (3 pairs), complex numbers with a negative real part (1 pair)), with an appropriate combination of κ_8 and κ_{SD}.

Possible adjustments In order to break time reversal symmetry of the evolution equations (Box 3.5), the possible simple adjustments are (1) to add $\mathcal{H}$, $\mathcal{S}$ or $\mathcal{G}^i$ terms to the equations of $\partial_t \phi$, $\partial_t \tilde{\gamma}_{ij}$, or $\partial_t \tilde{\Gamma}^i$, and/or (2) to add $\mathcal{M}_i$ or $\mathcal{A}$ terms to $\partial_t K$ or $\partial_t \tilde{A}_{ij}$. We write them generally, including the above proposal (B), as

$$\begin{aligned} \partial_t \phi &= \partial_t^B \phi + \kappa_{\phi\mathcal{H}}\, \alpha \mathcal{H} + \kappa_{\phi\mathcal{G}}\, \alpha \tilde{D}_k \mathcal{G}^k, && (3.57) \\ \partial_t \tilde{\gamma}_{ij} &= \partial_t^B \tilde{\gamma}_{ij} + \kappa_{SD}\, \alpha \tilde{\gamma}_{ij} \mathcal{H} + \kappa_{\tilde{\gamma}\mathcal{G}1}\, \alpha \tilde{\gamma}_{ij} \tilde{D}_k \mathcal{G}^k + \kappa_{\tilde{\gamma}\mathcal{G}2}\, \alpha \tilde{\gamma}_{k(i} \tilde{D}_{j)} \mathcal{G}^k \\ &\qquad + \kappa_{\tilde{\gamma}\mathcal{S}1}\, \alpha \tilde{\gamma}_{ij} \mathcal{S} + \kappa_{\tilde{\gamma}\mathcal{S}2}\, \alpha \tilde{D}_i \tilde{D}_j \mathcal{S}, && (3.58) \\ \partial_t K &= \partial_t^B K + \kappa_{K\mathcal{M}}\, \alpha \tilde{\gamma}^{jk} (\tilde{D}_j \mathcal{M}_k), && (3.59) \\ \partial_t \tilde{A}_{ij} &= \partial_t^B \tilde{A}_{ij} + \kappa_{A\mathcal{M}1}\, \alpha \tilde{\gamma}_{ij} (\tilde{D}^k \mathcal{M}_k) + \kappa_{A\mathcal{M}2}\, \alpha (\tilde{D}_{(i} \mathcal{M}_{j)}) \\ &\qquad + \kappa_{A\mathcal{A}1}\, \alpha \tilde{\gamma}_{ij} \mathcal{A} + \kappa_{A\mathcal{A}2}\, \alpha \tilde{D}_i \tilde{D}_j \mathcal{A}, && (3.60) \\ \partial_t \tilde{\Gamma}^i &= \partial_t^B \tilde{\Gamma}^i + \kappa_{\tilde{\Gamma}\mathcal{H}}\, \alpha \tilde{D}^i \mathcal{H} + \kappa_{\tilde{\Gamma}\mathcal{G}1}\, \alpha \mathcal{G}^i + \kappa_{\tilde{\Gamma}\mathcal{G}2}\, \alpha \tilde{D}^j \tilde{D}_j \mathcal{G}^i + \kappa_{\tilde{\Gamma}\mathcal{G}3}\, \alpha \tilde{D}^i \tilde{D}_j \mathcal{G}^j, && (3.61) \end{aligned}$$

where κs are possible multipliers (all $\kappa = 0$ reduce the system the standard BSSN system).

We show the effects of each terms in Table 4. The CAFs in the table are on the flat space background. We see several terms make negative CAFs, which might improve the stability than the previous system. For the readers convenience, we list up several best candidates here.

(D) A system which has 7 negative CAFs
Simply adding $\tilde{D}_{(i} \mathcal{M}_{j)}$ term to $\partial_t \tilde{A}_{ij}$ equation, say

$$\partial_t \tilde{A}_{ij} = \partial_t^{BSSN} \tilde{A}_{ij} + \kappa_{A\mathcal{M}2} \alpha (\tilde{D}_{(i} \mathcal{M}_{j)}) \tag{3.62}$$

with $\kappa_{A\mathcal{M}2} > 0$, CAFs on the flat background are 7 negative real CAFs.

	adjustment	CAFs	diag?	effect of the adjustment	
$\partial_t\phi$	$\kappa_{\phi\mathcal{H}}\,\alpha\mathcal{H}$	$(0,0,\pm\sqrt{-k^2}(*3),8\kappa_{\phi\mathcal{H}}k^2)$	no	$\kappa_{\phi\mathcal{H}}<0$ makes 1 Neg.	
$\partial_t\phi$	$\kappa_{\phi\mathcal{G}}\,\alpha\tilde{D}_k\mathcal{G}^k$	$(0,0,\pm\sqrt{-k^2}(*2)$, long expressions)	yes	$\kappa_{\phi\mathcal{G}}<0$ makes 2 Neg. 1 Pos.	
$\partial_t\tilde{\gamma}_{ij}$	$\kappa_{SD}\,\alpha\tilde{\gamma}_{ij}\mathcal{H}$	$(0,0,\pm\sqrt{-k^2}(*3),(3/2)\kappa_{SD}k^2)$	yes	$\kappa_{SD}<0$ makes 1 Neg.	Case (B)
$\partial_t\tilde{\gamma}_{ij}$	$\kappa_{\tilde{\gamma}\mathcal{G}1}\,\alpha\tilde{\gamma}_{ij}\tilde{D}_k\mathcal{G}^k$	$(0,0,\pm\sqrt{-k^2}(*2)$, long expressions)	yes	$\kappa_{\tilde{\gamma}\mathcal{G}1}>0$ makes 1 Neg.	
$\partial_t\tilde{\gamma}_{ij}$	$\kappa_{\tilde{\gamma}\mathcal{G}2}\,\alpha\tilde{\gamma}_{k(i}\tilde{D}_{j)}\mathcal{G}^k$	(0,0, long expressions)	yes	$\kappa_{\tilde{\gamma}\mathcal{G}2}<0$ makes 6 Neg. 1 Pos.	Case (E1)
$\partial_t\tilde{\gamma}_{ij}$	$\kappa_{\tilde{\gamma}\mathcal{S}1}\,\alpha\tilde{\gamma}_{ij}\mathcal{S}$	$(0,0,\pm\sqrt{-k^2}(*3),3\kappa_{\tilde{\gamma}\mathcal{S}1})$	no	$\kappa_{\tilde{\gamma}\mathcal{S}1}<0$ makes 1 Neg.	
$\partial_t\tilde{\gamma}_{ij}$	$\kappa_{\tilde{\gamma}\mathcal{S}2}\,\alpha\tilde{D}_i\tilde{D}_j\mathcal{S}$	$(0,0,\pm\sqrt{-k^2}(*3),-\kappa_{\tilde{\gamma}\mathcal{S}2}k^2)$	no	$\kappa_{\tilde{\gamma}\mathcal{S}2}>0$ makes 1 Neg.	
$\partial_t K$	$\kappa_{K\mathcal{M}}\,\alpha\tilde{\gamma}^{jk}(\tilde{D}_j\mathcal{M}_k)$	$(0,0,0,\pm\sqrt{-k^2}(*2)$, long expressions)	no	$\kappa_{K\mathcal{M}}<0$ makes 2 Neg.	
$\partial_t\tilde{A}_{ij}$	$\kappa_{A\mathcal{M}1}\,\alpha\tilde{\gamma}_{ij}(\tilde{D}^k\mathcal{M}_k)$	$(0,0,\pm\sqrt{-k^2}(*3),-\kappa_{A\mathcal{M}1}k^2)$	yes	$\kappa_{A\mathcal{M}1}>0$ makes 1 Neg.	
$\partial_t\tilde{A}_{ij}$	$\kappa_{A\mathcal{M}2}\,\alpha(\tilde{D}_{(i}\mathcal{M}_{j)})$	(0,0, long expressions)	yes	$\kappa_{A\mathcal{M}2}>0$ makes 7 Neg	Case (D)
$\partial_t\tilde{A}_{ij}$	$\kappa_{A\mathcal{A}1}\,\alpha\tilde{\gamma}_{ij}\mathcal{A}$	$(0,0,\pm\sqrt{-k^2}(*3),3\kappa_{A\mathcal{A}1})$	yes	$\kappa_{A\mathcal{A}1}<0$ makes 1 Neg.	
$\partial_t\tilde{A}_{ij}$	$\kappa_{A\mathcal{A}2}\,\alpha\tilde{D}_i\tilde{D}_j\mathcal{A}$	$(0,0,\pm\sqrt{-k^2}(*3),-\kappa_{A\mathcal{A}2}k^2)$	yes	$\kappa_{A\mathcal{A}2}>0$ makes 1 Neg.	
$\partial_t\tilde{\Gamma}^i$	$\kappa_{\tilde{\Gamma}\mathcal{H}}\,\alpha\tilde{D}^i\mathcal{H}$	$(0,0,\pm\sqrt{-k^2}(*3),-\kappa_{A\mathcal{A}2}k^2)$	no	$\kappa_{\tilde{\Gamma}\mathcal{H}}>0$ makes 1 Neg.	
$\partial_t\tilde{\Gamma}^i$	$\kappa_{\tilde{\Gamma}\mathcal{G}1}\,\alpha\mathcal{G}^i$	(0,0, long expressions)	yes	$\kappa_{\tilde{\Gamma}\mathcal{G}1}<0$ makes 6 Neg. 1 Pos.	Case (E2)
$\partial_t\tilde{\Gamma}^i$	$\kappa_{\tilde{\Gamma}\mathcal{G}2}\,\alpha\tilde{D}^j\tilde{D}_j\mathcal{G}^i$	(0,0, long expressions)	yes	$\kappa_{\tilde{\Gamma}\mathcal{G}2}>0$ makes 2 Neg. 1 Pos.	
$\partial_t\tilde{\Gamma}^i$	$\kappa_{\tilde{\Gamma}\mathcal{G}3}\,\alpha\tilde{D}^i\tilde{D}_j\mathcal{G}^j$	(0,0, long expressions)	yes	$\kappa_{\tilde{\Gamma}\mathcal{G}3}>0$ makes 2 Neg. 1 Pos.	

Table 4: Possible adjustements which make a real-part constraint amplification factors negative. The column of adjustments are nonzero multipliers in terms of (3.57)-(3.61), which all violate time reversal symmetry of the equation. The column 'diag?' indicates diagonalizability of the constraint propagation matrix. Neg./Pos. means negative/positive respectively.

(E) A system which has 6 negative and 1 positive CAFs
The below two adjustments will make 6 negative real CAFs, while they also produce one positive real CAF (a constraint violating mode). The effectiveness is not clear at this moment, but we think they are worth to be tested in numerical experiments.

$$(E1) \qquad \partial_t\tilde{\gamma}_{ij}=\partial_t^{BSSN}\tilde{\gamma}_{ij}+\kappa_{\tilde{\gamma}\mathcal{G}2}\,\alpha\tilde{\gamma}_{k(i}\tilde{D}_{j)}\mathcal{G}^k, \qquad \text{with} \qquad \kappa_{\tilde{\gamma}\mathcal{G}2}<0. \tag{3.63}$$

$$(E2) \qquad \partial_t\tilde{\Gamma}^i=\partial_t^{BSSN}\tilde{\Gamma}^i+\kappa_{\tilde{\Gamma}\mathcal{G}2}\,\alpha\tilde{D}^j\tilde{D}_j\mathcal{G}^i, \qquad \text{with} \qquad \kappa_{\tilde{\Gamma}\mathcal{G}2}<0. \tag{3.64}$$

3.3.4 Remarks

We have studied step-by-step where the replacements in the equations affect and/or newly added constraints work, by checking whether the error of constraints (if it exists) will decay or propagate away. Alcubierre et al [5] pointed out the importance of the replacement (adjustment) of terms in the evolution equation due to the momentum constraint, and our analysis clearly explain why they concluded this is the key. Not only this adjustment, we found, but also other adjustments and other introductions of new constraints also contribute to making the evolution system more stable. We found that if we missed a particular adjustment, then the expected stability behaviour occationally gets worse than the ADM system. We further propose other adjustments of the set of equations which may have better features for numerical treatments.

The discussion was only in the flat background spacetime, and further analysis is in progress. However, we rather believe that the general fundamental aspects of constraint propagation analysis are already revealed here. This is because, for the ADM and its adjusted formulation cases, we found that the better formulations in the flat background are also better in the Schwarzschild spacetime, while there are differences on the effective adjusting multipliers or the effective coordinate ranges [78, 95]. Actually, recently Yo, Baumgarte and Shapiro [90] reported their simulations of stationary rotating black hole, and mentioned that the above proposal (B) was contributed to maintain their evolution of Kerr black hole (J/M up to $0.9M$) for long time ($t\sim 6000M$). Their results also indicates that the evolved solution is closed to the exact one, that is, the constrained surface.

4 Outlook

4.1 What we have achieved

Let us summarize our story first. The beginnings of the study are:

- General relativistic numerical simulations are quite important in astrophysical studies, but we do not have a definite recipe to integrate the Einstein equations in a long-term stable and accurate manner.

- The most standard approach is to decompose the space-time into 3-space and time (3 + 1 decomposition), to solve the constraints for obtaining the initial data, and to evolve the space-time applying a *free evolution* scheme.

- Over a period of decades, the community has been observing the violation of constraints in their simulations, and their consensus is that we have to find a better formulation of the Einstein evolution equations.

We reviewed recent efforts on this problem by categorizing them into
(0) The standard ADM formulation (§2.0, Box 2.1),
(1) The modified ADM (so-called BSSN) formulation (§2.1, Box 2.3),
(2) Hyperbolic formulations (§2.2, Box 2.5), and
(3) Asymptotically constrained formulations (§2.3).
Among them, the approach (2) is perhaps best justified on mathematical grounds. However, as we critically reviewed in §2.2.3, the practical advantages may not be available unless we kill the lower-order terms in its hyperbolized equation, as in KST's formulation (Box 2.6).

We therefore proceeded in the direction (3). Our approach, which we term *adjusted system*, is to construct a system that has its constraint surface as an attractor. Our unified view is to understand the evolution system by evaluating its constraint propagation. Especially we proposed to analyze the constraint amplification factors (Box 3.1) which are the eigenvalues of the homogenized constraint propagation equations. We analyzed the system based on our conjecture (Box 3.2/3.3) whether the constraint amplification factors suggest the constraint to decay/propagate or not. We concluded that

- The constraint propagation features become different by simply adding constraint terms to the original evolution equations (we call this the *adjustment* of the evolution equations).

- There *is* a constraint-violating mode in the standard ADM evolution system when we apply it to a single non-rotating black hole space-time, and its growth rate is larger near the black-hole horizon.

- Such a constraint-violating mode can be killed if we adjust the evolution equations with a particular modification using constraint terms (Box 2.7). An effective guideline is to adjust terms as they break the time-reversal symmetry of the equations (Box 3.5).

- Our expectations are borne out in simple numerical experiments using the Maxwell, Ashtekar, and ADM systems. However the modifications are not yet perfect to prevent non-linear growth of the constraint violation.

- We understand why the BSSN formulation works better than the ADM one in the limited case (perturbative analysis in the flat background), and further we proposed modified evolution equations along the lines of our previous procedure.

The common key to the problem is how to adjust the evolution equations with constraints. Any adjusted systems are mathematically equivalent if the constraints are completely satisfied, but this is not the case for numerical simulations. Replacing terms with constraints is one of the normal steps when people hyperbolize equations. Our approach is to employ the evaluation process of constraint amplification factors for an alternative guideline to hyperbolization of the system.

4.2 Next steps

Here are some directions for future researches in this formulation problem of the Einstein equations.

Generalize the adjusted systems: We have tried to unify all the efforts of the community applying the idea of "adjusted systems". Our newly modified equations are working as desired up to the current numerical tests, but we see also that the effect is not yet perfect to remove non-linear error growth that terminates numerical simulations. We suspect that this is due to the current eigenvalue analysis based on a perturbative analysis with fixing adjusting multiplier, κ. One remedy is to generalize the determination process of κ, say, dynamically and automatically under a suitable principle. We are now working on a method to control the violation of *each* constraint independently, or an additional supporting mathematical criteria to realize more robust stabilizations.

More on hyperbolic formulations: We have already pointed out several directions on hyperbolic efforts in §2.2.3. We here only point out the links to the initial-boundary value problem (IBVP). In order to avoid unphysical incoming information from the boundary of computation, a proper treatment of the boundary is quite important problem in numerical simulations. If we can treat boundary condition as a part of mathematical framework, that would be a great step to promote researches. The IBVP approach to the Einstein equations is quite new and has only been studied in a restricted cases. The current proposals are based on a particular symmetric hyperbolic formulation or under a certain assumption to the symmetry of space-time. More general treatments on IBVP are expected.

Alternative new ideas?: We have to be open to develop an alternative approach to this formulation problem. If our goal is to obtain a stable system on its constrained surface, then there may be at least three fundamental approaches: (1) construct an improved evolution system (as we discussed most), (2) develop a "maintenance method" for the system, or (3) redefine or classify the stability of the system. An idea of automatic adjustment of multipliers in adjusted system is a sort of "maintenance". We guess there might be a key in existing theories such as control theories, optimization methods (convex functional theories), mathematical programming methods, or others.

Numerical comparisons of formulations: Fortunately, an effort toward systematic numerical comparisons of different formulations of the Einstein equations has been organized and started recently. The workshop "Comparisons of Formulations of Einstein's equations for Numerical Relativity" was held at Mexico City in May 2002 [5], and more than 20 people attended this 2-week workshop. This project intends to provide a format for comparisons in a common numerical framework. That is, comparing different formulations using the same initial data, the same resolution, the same integration scheme, the same boundary treatment, and the same output. Comparisons are now in progress for vacuum and regular space-time, and we hope to extend them to black-hole space-time next. The first reports are in preparation [59].

4.3 Final remarks

If we say the final goal of this project is to find a robust algorithm to obtain long-term accurate and stable time-evolution method, then the recipe should be a combination of (a) formulations of the

[5] Follow-up information is available at `http://www.nuclecu.unam.mx/~gravit/main_rn.html`.

evolution equations, (b) choice of gauge conditions, (c) treatment of boundary conditions, and (d) numerical integration methods. We are in the stages of solving this mixed puzzle. The ideal almighty algorithm may not exit, but we believe our accumulating experience will make the ones we do have more robust and automatic.

We have written this review from the viewpoint that the general relativity is a constrained dynamical system. This is not only a proper problem in general relativity, but also in many physical systems such as electrodynamics, magnetohydrodynamics, molecular dynamics, mechanical dynamics, and so on. Therefore sharing the thoughts between different field will definitely accelerate the progress.

When we discuss asymptotically constrained manifolds, we implicitly assume that the dynamics could be expressed on a wider manifold. Recently we found that such a proposal is quite similar with techniques in e.g. molecular dynamics. For example, people assume an extended environment when simulating molecular dynamics under constant pressure (with a potential piston) or a constant temperature (with a potential thermostat) (see e.g. [65]). We have also noticed that a dynamical adjusting method of Lagrange multipliers has been developed in multi-body mechanical dynamics (see e.g. [62]). We are now trying to apply these ideas back into numerical relativity.

In such a way, communication and interaction between different fields is encouraged. Cooperation between numerical and mathematical scientists is necessary. By interchanging ideas, we hope we will reach our goal in next few years, and obtain interesting physical results and predictions. It is our personal view that exciting revolutions in numerical relativity are coming soon.

A General expressions of ADM constraint propagation equations

For the reader's convenience, we express here the constraint propagation equations generally, considering the adjustments to the evolution equations.

A.1 The standard ADM equations and constraint propagations

We start by analyzing the standard ADM system, that is, with evolution equations (2.5) and (2.6) and constraint equations (2.7) and (2.8).

The constraint propagation equations, which are the time evolution equations of the Hamiltonian constraint (2.7) and the momentum constraints (2.8).

Expression using $\mathcal{H}$ and $\mathcal{M}_i$ The constraint propagation equations can be written as [these are the same with (2.9) and (2.10)]

$$\begin{aligned}
\partial_t \mathcal{H} &= \beta^j(\partial_j \mathcal{H}) + 2\alpha K \mathcal{H} - 2\alpha \gamma^{ij}(\partial_i \mathcal{M}_j) \\
&\quad + \alpha(\partial_l \gamma_{mk})(2\gamma^{ml}\gamma^{kj} - \gamma^{mk}\gamma^{lj})\mathcal{M}_j - 4\gamma^{ij}(\partial_j \alpha)\mathcal{M}_i, \qquad (A.1)\\
\partial_t \mathcal{M}_i &= -(1/2)\alpha(\partial_i \mathcal{H}) - (\partial_i \alpha)\mathcal{H} + \beta^j(\partial_j \mathcal{M}_i) \\
&\quad + \alpha K \mathcal{M}_i - \beta^k \gamma^{jl}(\partial_i \gamma_{lk})\mathcal{M}_j + (\partial_i \beta_k)\gamma^{kj}\mathcal{M}_j. \qquad (A.2)
\end{aligned}$$

This is a suitable form to discuss hyperbolicity of the system. The simplest derivation of (A.1) and (A.2) is by using the Bianchi identity, which can be seen in Frittelli [39].

A shorter expression is available, e.g.

$$\begin{aligned}
\partial_t \mathcal{H} &= \beta^l \partial_l \mathcal{H} + 2\alpha K \mathcal{H} - 2\alpha \gamma^{-1/2}\partial_l(\sqrt{\gamma}\mathcal{M}^l) - 4(\partial_l \alpha)\mathcal{M}^l \\
&= \beta^l \nabla_l \mathcal{H} + 2\alpha K \mathcal{H} - 2\alpha(\nabla_l \mathcal{M}^l) - 4(\nabla_l \alpha)\mathcal{M}^l, \qquad (A.3)\\
\partial_t \mathcal{M}_i &= -(1/2)\alpha(\partial_i \mathcal{H}) - (\partial_i \alpha)\mathcal{H} + \beta^l \nabla_l \mathcal{M}_i + \alpha K \mathcal{M}_i + (\nabla_i \beta_l)\mathcal{M}^l \\
&= -(1/2)\alpha(\nabla_i \mathcal{H}) - (\nabla_i \alpha)\mathcal{H} + \beta^l \nabla_l \mathcal{M}_i + \alpha K \mathcal{M}_i + (\nabla_i \beta_l)\mathcal{M}^l, \qquad (A.4)
\end{aligned}$$

or by using Lie derivatives along αn^μ,

$$\begin{aligned}
\pounds_{\alpha n^\mu} \mathcal{H} &= 2\alpha K \mathcal{H} - 2\alpha \gamma^{-1/2}\partial_l(\sqrt{\gamma}\mathcal{M}^l) - 4(\partial_l \alpha)\mathcal{M}^l, \qquad (A.5)\\
\pounds_{\alpha n^\mu} \mathcal{M}_i &= -(1/2)\alpha(\partial_i \mathcal{H}) - (\partial_i \alpha)\mathcal{H} + \alpha K \mathcal{M}_i. \qquad (A.6)
\end{aligned}$$

Expression using γ_{ij} and K_{ij} In order to check the effects of the adjustments in (2.5) and (2.6) to constraint propagation, it is useful to re-express (A.1) and (A.2) using γ_{ij} and K_{ij}. By a straightforward calculation, we obtain an expression as

$$\begin{aligned}
\partial_t \mathcal{H} &= H_1^{mn}(\partial_t \gamma_{mn}) + H_2^{imn}\partial_i(\partial_t \gamma_{mn}) + H_3^{ijmn}\partial_i \partial_j(\partial_t \gamma_{mn}) + H_4^{mn}(\partial_t K_{mn}), \qquad (A.7)\\
\partial_t \mathcal{M}_i &= M_{1i}{}^{mn}(\partial_t \gamma_{mn}) + M_{2i}{}^{jmn}\partial_j(\partial_t \gamma_{mn}) + M_{3i}{}^{mn}(\partial_t K_{mn}) + M_{4i}{}^{jmn}\partial_j(\partial_t K_{mn}), \qquad (A.8)
\end{aligned}$$

where

$$\begin{aligned}
H_1^{mn} &:= -2R^{(3)mn} - \Gamma^p_{kj}\Gamma^k_{pi}\gamma^{mi}\gamma^{nj} + \Gamma^m\Gamma^n \\
&\quad + \gamma^{ij}\gamma^{np}(\partial_i \gamma^{mk})(\partial_j \gamma_{kp}) - \gamma^{mp}\gamma^{ni}(\partial_i \gamma^{kj})(\partial_j \gamma_{kp}) - 2KK^{mn} + 2K^n{}_j K^{mj}, \qquad (A.9)\\
H_2^{imn} &:= -2\gamma^{mi}\Gamma^n - (3/2)\gamma^{ij}(\partial_j \gamma^{mn}) + \gamma^{mj}(\partial_j \gamma^{in}) + \gamma^{mn}\Gamma^i, \qquad (A.10)\\
H_3^{ijmn} &:= -\gamma^{ij}\gamma^{mn} + \gamma^{in}\gamma^{mj}, \qquad (A.11)\\
H_4^{mn} &:= 2(K\gamma^{mn} - K^{mn}), \qquad (A.12)\\
M_{1i}{}^{mn} &:= \gamma^{nj}(\partial_i K^m{}_j) - \gamma^{mj}(\partial_j K^n{}_i) + (1/2)(\partial_j \gamma^{mn})K^j{}_i + \Gamma^n K^m{}_i, \qquad (A.13)
\end{aligned}$$

$$
\begin{aligned}
M_{2i}{}^{jmn} &:= -\gamma^{mj}K^n{}_i + (1/2)\gamma^{mn}K^j{}_i + (1/2)K^{mn}\delta_i^j, & \text{(A.14)}\\
M_{3i}{}^{mn} &:= -\delta_i^n\Gamma^m - (1/2)(\partial_i\gamma^{mn}), & \text{(A.15)}\\
M_{4i}{}^{jmn} &:= \gamma^{mj}\delta_i^n - \gamma^{mn}\delta_i^j, & \text{(A.16)}
\end{aligned}
$$

where we expressed $\Gamma^m = \Gamma^m_{ij}\gamma^{ij}$.

A.2 Constraint propagations for the adjusted ADM systems

Generally, we here write the adjustment terms to (2.5) and (2.6) using (2.7) and (2.8) by the following combinations, [these are the same with (3.7) and (3.8)]

$$
\begin{aligned}
&\text{adjustment term of} \quad \partial_t\gamma_{ij}: \quad && +P_{ij}\mathcal{H} + Q^k{}_{ij}\mathcal{M}_k + p^k{}_{ij}(\nabla_k\mathcal{H}) + q^{kl}{}_{ij}(\nabla_k\mathcal{M}_l), & \text{(A.17)}\\
&\text{adjustment term of} \quad \partial_t K_{ij}: \quad && +R_{ij}\mathcal{H} + S^k{}_{ij}\mathcal{M}_k + r^k{}_{ij}(\nabla_k\mathcal{H}) + s^{kl}{}_{ij}(\nabla_k\mathcal{M}_l), & \text{(A.18)}
\end{aligned}
$$

where P, Q, R, S and p, q, r, s are multipliers (please do not confuse R_{ij} with three Ricci curvature that we write as $R^{(3)}_{ij}$). We adjust them only using up to the first derivatives in order to make the discussion simple.

By substituting the above adjustments into (A.7) and (A.8), we can write the adjusted constraint propagation equations as

$$
\begin{aligned}
\partial_t\mathcal{H} &= \text{(original terms)}\\
&\quad +H_1^{mn}[P_{mn}\mathcal{H} + Q^k_{mn}\mathcal{M}_k + p^k{}_{mn}(\nabla_k\mathcal{H}) + q^{kl}{}_{mn}(\nabla_k\mathcal{M}_l)]\\
&\quad +H_2^{imn}\partial_i[P_{mn}\mathcal{H} + Q^k_{mn}\mathcal{M}_k + p^k{}_{mn}(\nabla_k\mathcal{H}) + q^{kl}{}_{mn}(\nabla_k\mathcal{M}_l)]\\
&\quad +H_3^{ijmn}\partial_i\partial_j[P_{mn}\mathcal{H} + Q^k_{mn}\mathcal{M}_k + p^k{}_{mn}(\nabla_k\mathcal{H}) + q^{kl}{}_{mn}(\nabla_k\mathcal{M}_l)]\\
&\quad +H_4^{mn}[R_{mn}\mathcal{H} + S^k_{mn}\mathcal{M}_k + r^k{}_{mn}(\nabla_k\mathcal{H}) + s^{kl}{}_{mn}(\nabla_k\mathcal{M}_l)], & \text{(A.19)}\\
\partial_t\mathcal{M}_i &= \text{(original terms)}\\
&\quad +M_{1i}{}^{mn}[P_{mn}\mathcal{H} + Q^k_{mn}\mathcal{M}_k + p^k{}_{mn}(\nabla_k\mathcal{H}) + q^{kl}{}_{mn}(\nabla_k\mathcal{M}_l)]\\
&\quad +M_{2i}{}^{jmn}\partial_j[P_{mn}\mathcal{H} + Q^k_{mn}\mathcal{M}_k + p^k{}_{mn}(\nabla_k\mathcal{H}) + q^{kl}{}_{mn}(\nabla_k\mathcal{M}_l)]\\
&\quad +M_{3i}{}^{mn}[R_{mn}\mathcal{H} + S^k_{mn}\mathcal{M}_k + r^k{}_{mn}(\nabla_k\mathcal{H}) + s^{kl}{}_{mn}(\nabla_k\mathcal{M}_l)]\\
&\quad +M_{4i}{}^{jmn}\partial_j[R_{mn}\mathcal{H} + S^k_{mn}\mathcal{M}_k + r^k{}_{mn}(\nabla_k\mathcal{H}) + s^{kl}{}_{mn}(\nabla_k\mathcal{M}_l)]. & \text{(A.20)}
\end{aligned}
$$

Here the "original terms" can be understood either as (A.1) and (A.2), or as (A.7) and (A.8). Therefore, for example, we can see that adjustments to $\partial_t\gamma_{ij}$ do not always keep the constraint propagation equations in the first order form, due to their contribution in the third adjusted term in (A.19).

We note that these expressions of constraint propagation equations are equivalent when we include the cosmological constant and/or matter terms.

B Numerical demonstrations using the Ashtekar formulation

This appendix is devoted to introduce our numerical comparisons between three levels of hyperbolicity using Ashtekar variables. Details are available in [77, 94].

B.1 The Ashtekar formulation

The key feature of Ashtekar's formulation of general relativity [13] is the introduction of a self-dual connection as one of the basic dynamical variables. Let us write the metric $g_{\mu\nu}$ using the tetrad E^I_μ as $g_{\mu\nu} = E^I_\mu E^J_\nu \eta_{IJ}$ [6]. Define its inverse, E^μ_I, by $E^\mu_I := E^J_\nu g^{\mu\nu}\eta_{IJ}$ and we impose $E^0_a = 0$ as the gauge condition. We define SO(3,C) connections ${}^\pm\mathcal{A}^a_\mu := \omega^{0a}_\mu \mp (i/2)\epsilon^a{}_{bc}\,\omega^{bc}_\mu$, where ω^{IJ}_μ is a spin connection 1-form (Ricci connection), $\omega^{IJ}_\mu := E^{I\nu}\nabla_\mu E^J_\nu$. Ashtekar's plan is to use only the self-dual part of the connection ${}^+\mathcal{A}^a_\mu$ and to use its spatial part ${}^+\mathcal{A}^a_i$ as a dynamical variable. Hereafter, we simply denote ${}^+\mathcal{A}^a_\mu$ as $\mathcal{A}^a_\mu$.

The lapse function, N, and shift vector, N^i, both of which we treat as real-valued functions[7], are expressed as $E^\mu_0 = (1/N, -N^i/N)$. This allows us to think of E^μ_0 as a normal vector field to Σ spanned by the condition $t = x^0$ =const., which plays the same role as that of the ADM formulation. Ashtekar treated the set $(\tilde{E}^i_a,\ \mathcal{A}^a_i)$ as basic dynamical variables, where $\tilde{E}^i_a$ is an inverse of the densitized triad defined by $\tilde{E}^i_a := eE^i_a$, where $e := \det E^a_i$ is a density [8] . This pair forms the canonical set. In the case of pure gravitational spacetime, the Hilbert action takes the form

$$S = \int \mathrm{d}^4x[(\partial_t\mathcal{A}^a_i)\tilde{E}^i_a + (i/2)\underset{\sim}{N}\tilde{E}^i_a\tilde{E}^j_b F^c_{ij}\epsilon^{ab}{}_c - N^i F^a_{ij}\tilde{E}^j_a + \mathcal{A}^a_0\,\mathcal{D}_i\tilde{E}^i_a], \qquad \text{(B.21)}$$

where $\underset{\sim}{N} := e^{-1}N$, $F^a_{\mu\nu} := 2\partial_{[\mu}\mathcal{A}^a_{\nu]} - i\epsilon^a{}_{bc}\,\mathcal{A}^b_\mu\mathcal{A}^c_\nu$ is the curvature 2-form, $\mathcal{D}_i\tilde{E}^j_a := \partial_i\tilde{E}^j_a - i\epsilon_{ab}{}^c\,\mathcal{A}^b_i\tilde{E}^j_c$.

The action (B.21) gives us the following evolution equations and constraints:

The Ashtekar formulation [13]: **Box B.1**

The dynamical variables are $(\tilde{E}^i_a, \mathcal{A}^a_i)$.

The evolution equations for a set of $(\tilde{E}^i_a, \mathcal{A}^a_i)$ are

$$\partial_t\tilde{E}^i_a = -i\mathcal{D}_j(\epsilon^{cb}{}_a\,\underset{\sim}{N}\tilde{E}^j_c\tilde{E}^i_b) + 2\mathcal{D}_j(N^{[j}\tilde{E}^{i]}_a) + i\mathcal{A}^b_0\epsilon_{ab}{}^c\,\tilde{E}^i_c, \qquad \text{(B.22)}$$

$$\partial_t\mathcal{A}^a_i = -i\epsilon^{ab}{}_c\,\underset{\sim}{N}\tilde{E}^j_b F^c_{ij} + N^j F^a_{ji} + \mathcal{D}_i\mathcal{A}^a_0, \qquad \text{(B.23)}$$

where $\mathcal{D}_j X^{ji}_a := \partial_j X^{ji}_a - i\epsilon_{ab}{}^c\mathcal{A}^b_j X^{ji}_c$, and $F^a_{ij} := 2\partial_{[i}\mathcal{A}^a_{j]} - i\epsilon^a{}_{bc}\,\mathcal{A}^b_i\mathcal{A}^c_j$.

Constraint equations: (Hamiltonian, momentum and Gauss constraints)

$$\mathcal{C}^{\mathrm{ASH}}_H := (i/2)\epsilon^{ab}{}_c\,\tilde{E}^i_a\tilde{E}^j_b F^c_{ij} \approx 0, \qquad \text{(B.24)}$$

$$\mathcal{C}^{\mathrm{ASH}}_{Mi} := -F^a_{ij}\tilde{E}^j_a \approx 0, \qquad \text{(B.25)}$$

$$\mathcal{C}^{\mathrm{ASH}}_{Ga} := \mathcal{D}_i\tilde{E}^i_a \approx 0, \qquad \text{(B.26)}$$

---

[6] We use $I, J = (0), \cdots, (3)$ and $a, b = (1), \cdots, (3)$ are $SO(1,3)$, $SO(3)$ indices respectively. We raise and lower $\mu, \nu, \cdots$ by $g^{\mu\nu}$ and $g_{\mu\nu}$ (the Lorentzian metric); $I, J, \cdots$ by $\eta^{IJ} = \mathrm{diag}(-1,1,1,1)$ and η_{IJ}; $i, j, \cdots$ by γ^{ij} and γ_{ij} (the three-metric); $a, b, \cdots$ by δ^{ab} and δ_{ab}. We also use volume forms ϵ_{abc}: $\epsilon_{abc}\epsilon^{abc} = 3!$.

[7] N and N^i are the same with α and β^i in the previous text. We follow this conventional notation in this appendix.

[8] For later convenience, $e^2 = \det\tilde{E}^i_a = (\det E^a_i)^2 = (1/6)\epsilon^{abc}\underset{\sim}{\epsilon}_{ijk}\tilde{E}^i_a\tilde{E}^j_b\tilde{E}^k_c$, where $\epsilon_{ijk} := \epsilon_{abc}E^a_iE^b_jE^c_k$ and $\underset{\sim}{\epsilon}_{ijk} := e^{-1}\epsilon_{ijk}$. When $(i,j,k) = (1,2,3)$, we have $\epsilon_{ijk} = e$, $\underset{\sim}{\epsilon}_{ijk} = 1$, $\epsilon^{ijk} = e^{-1}$, and $\tilde{\epsilon}^{ijk} = 1$.

We have to consider the reality conditions when we use this formalism to describe the classical Lorentzian spacetime. Fortunately, the metric will remain on its real-valued constraint surface during time evolution automatically if we prepare initial data which satisfies the reality condition. More practically, we further require that triad is real-valued. But again this reality condition appears as a gauge restriction on $\mathcal{A}^a_0$[91], which can be imposed at every time step. In our actual simulation, we prepare our initial data using the standard ADM approach, so that we have no difficulties in maintaining these reality conditions.

B.2 Reformulate the Ashtekar evolution equations

B.2.1 Strongly and Symmetric Hyperbolic systems

The authors' recent studies showed the following:

(a) The original set of dynamical equations (B.22) and (B.23) [the *original* equations] already forms a weakly hyperbolic system [93]. So that we regard the mathematical structure of the original equations as one step advanced from the standard ADM.

(b) Further, we can construct higher levels of hyperbolic systems by restricting the gauge condition and/or by adding constraint terms, $\mathcal{C}_H^{\rm ASH}, \mathcal{C}_{Mi}^{\rm ASH}$ and $\mathcal{C}_{Ga}^{\rm ASH}$, to the original equations.

- by requiring additional gauge conditions *or* adding constraints to the dynamical equations, we can obtain a strongly hyperbolic system [93],
- by requiring additional gauge conditions *and* adding constraints to the dynamical equations, we can obtain a symmetric hyperbolic system [92, 93].

(c) Based on the above symmetric hyperbolic system, we can construct an Ashtekar version of the λ-system [24] which is robust against perturbative errors for both constraints and reality conditions [76].

In order to obtain a symmetric hyperbolic system [9], we add constraint terms to the right-hand-side of (B.22) and (B.23). The adjusted dynamical equations,

$$\begin{aligned}
\partial_t \tilde{E}^i_a &= -i\mathcal{D}_j(\epsilon^{cb}{}_a \underset{\sim}{N} \tilde{E}^j_c \tilde{E}^i_b) + 2\mathcal{D}_j(N^{[j}\tilde{E}^{i]}_a) + i\mathcal{A}^b_0 \epsilon_{ab}{}^c \tilde{E}^i_c + \kappa_1 P^i{}_{ab}\mathcal{C}_G^{{\rm ASH}\,b}, && \text{(B.27)}\\
&\quad \text{where} \qquad P^i{}_{ab} \equiv N^i \delta_{ab} + i\underset{\sim}{N}\epsilon_{ab}{}^c \tilde{E}^i_c, \\
\partial_t \mathcal{A}^a_i &= -i\epsilon^{ab}{}_c \underset{\sim}{N} \tilde{E}^j_b F^c_{ij} + N^j F^a_{ji} + \mathcal{D}_i \mathcal{A}^a_0 + \kappa_2 Q^a_i \mathcal{C}_H^{\rm ASH} + \kappa_3 R_i{}^{ja}\mathcal{C}_{Mj}^{\rm ASH}, && \text{(B.28)}\\
&\quad \text{where} \qquad Q^a_i \equiv e^{-2}\underset{\sim}{N}\tilde{E}^a_i, \qquad R_i{}^{ja} \equiv ie^{-2}\underset{\sim}{N}\epsilon^{ac}{}_b \tilde{E}^b_i \tilde{E}^j_c
\end{aligned}$$

form a symmetric hyperbolicity if we further require $\kappa_1 = \kappa_2 = \kappa_3 = 1$ and the gauge conditions,

$$\mathcal{A}^a_0 = \mathcal{A}^a_i N^i, \qquad \partial_i N = 0. \tag{B.29}$$

We remark that the adjusted coefficients, $P^i{}_{ab}, Q^a_i, R_i{}^{ja}$, for constructing the symmetric hyperbolic system are uniquely determined, and there are no other additional terms (say, no $\mathcal{C}_H^{\rm ASH}, \mathcal{C}_M^{\rm ASH}$ for $\partial_t \tilde{E}^i_a$, no $\mathcal{C}_G^{\rm ASH}$ for $\partial_t \mathcal{A}^a_i$) [93]. The gauge conditions, (B.29), are consequences of the consistency with (triad) reality conditions.

We can also construct a strongly (or diagonalizable) hyperbolic system by restricting to a gauge $N^l \neq 0, \pm N\sqrt{\gamma^{ll}}$ (where γ^{ll} is the three-metric and we do not sum indices here) for the original

[9] Iriondo et al [48] presented a symmetric hyperbolic expression in a different form. The differences between ours and theirs are discussed in [92, 93].

	system	variables	Eqs of motion	remark
I	Ashtekar (weakly hyp.)	$(\tilde{E}^i_a, \mathcal{A}^a_i)$	(B.22), (B.23) (original)	"original" eqs.
II	Ashtekar (strongly hyp.)	$(\tilde{E}^i_a, \mathcal{A}^a_i)$	(B.27), (B.28) (with $\kappa = 1$)	(B.30) required
III	Ashtekar (symmetric hyp.)	$(\tilde{E}^i_a, \mathcal{A}^a_i)$	(B.27), (B.28) (with $\kappa = 1$)	(B.29) required
adj	Ashtekar (adjusted)	$(\tilde{E}^i_a, \mathcal{A}^a_i)$	(B.27), (B.28) (with $\kappa \neq 1$)	
λ	Ashtekar-λ-system	$(\tilde{E}^i_a, \mathcal{A}^a_i, \lambda, \lambda_i, \lambda_a)$	(B.37)	controls $\mathcal{C}_H, \mathcal{C}_{Mi}, \mathcal{C}_{Ga}$

Table 5: List of systems that we compare in this appendix.

equations (B.22), (B.23). Or we can also construct from the adjusted equations, (B.27) and (B.28), together with the gauge condition

$$\mathcal{A}^a_0 = \mathcal{A}^a_i N^i. \tag{B.30}$$

As for the strongly hyperbolic system, we hereafter take the latter expression.

In Table 5, we summarized the equations to be used for our comparisons.

B.2.2 Ashtekar-λ-system

In §2.3.1, we introduced an idea to construct a robust evolution system against a perturbative error, named "λ-system" [24]. Based on the above symmetric hyperbolic equations, we constructed the Ashtekar version of the "λ-system" [76]. Here we present our system which evolves the spacetime to the constraint surface, $\mathcal{C}_H \approx \mathcal{C}_{Mi} \approx \mathcal{C}_{Ga} \approx 0$ as the attractor. In [76], we also presented a system which controls the perturbative violation of the reality condition.

We introduce new variables $(\lambda, \lambda_i, \lambda_a)$, as they obey the dissipative evolution equations

$$\begin{aligned}
\partial_t \lambda &= \alpha_1 \, \mathcal{C}_H - \beta_1 \, \lambda, && \text{(B.31)}\\
\partial_t \lambda_i &= \alpha_2 \, \tilde{\mathcal{C}}_{Mi} - \beta_2 \, \lambda_i, && \text{(B.32)}\\
\partial_t \lambda_a &= \alpha_3 \, \mathcal{C}_{Ga} - \beta_3 \, \lambda_a, && \text{(B.33)}
\end{aligned}$$

where $\alpha_i \neq 0$ (allowed to be complex numbers) and $\beta_i > 0$ (real numbers) are constants.

If we take $u^{(DL)}_\alpha = (\tilde{E}^i_a, \mathcal{A}^a_i, \lambda, \lambda_i, \lambda_a)$ as a set of dynamical variables, then the principal part of (B.31)-(B.33) can be written as

$$\begin{aligned}
\partial_t \lambda &\cong -i\alpha_1 \epsilon^{bcd} \tilde{E}^j_c \tilde{E}^l_d (\partial_l \mathcal{A}^b_j), && \text{(B.34)}\\
\partial_t \lambda_i &\cong \alpha_2[-e\delta^l_i \tilde{E}^j_b + e\delta^j_i \tilde{E}^l_b](\partial_l \mathcal{A}^b_j), && \text{(B.35)}\\
\partial_t \lambda_a &\cong \alpha_3 \partial_l \tilde{E}^l_a. && \text{(B.36)}
\end{aligned}$$

The characteristic matrix of the system $u^{(DL)}_\alpha$ does not form a Hermitian matrix. However, if we modify the right-hand-side of the evolution equation of $(\tilde{E}^i_a, \mathcal{A}^a_i)$, then the set becomes a symmetric hyperbolic system. This is done by adding $\bar{\alpha}_3 \gamma^{il}(\partial_l \lambda_a)$ to the equation of $\partial_t \tilde{E}^i_a$, and by adding $i\bar{\alpha}_1 \epsilon^a{}_c{}^d \tilde{E}^c_i \tilde{E}^l_d (\partial_l \lambda) + \bar{\alpha}_2(-e\gamma^{lm} \tilde{E}^a_i + e\delta^m_i \tilde{E}^{la})(\partial_l \lambda_m)$ to the equation of $\partial_t \mathcal{A}^a_i$. The final principal part, then, is written as

$$\partial_t \begin{pmatrix} \tilde{E}^i_a \\ \mathcal{A}^a_i \\ \lambda \\ \lambda_i \\ \lambda_a \end{pmatrix} \cong \begin{pmatrix} A^l{}_a{}^{bi}{}_m & 0 & 0 & 0 & \bar{\alpha}_3 \gamma^{il} \delta_a{}^b \\ 0 & D^{la}{}_{ib}{}^m & i\bar{\alpha}_1 \epsilon^a{}_c{}^d \tilde{E}^c_i \tilde{E}^l_d & \bar{\alpha}_2 e(\delta^m_i \tilde{E}^{la} - \gamma^{lm} \tilde{E}^a_i) & 0 \\ 0 & -i\alpha_1 \epsilon_b{}^{cd} \tilde{E}^m_c \tilde{E}^l_d & 0 & 0 & 0 \\ 0 & \alpha_2 e(\delta^m_i \tilde{E}^l_b - \delta^l_i \tilde{E}^m_b) & 0 & 0 & 0 \\ \alpha_3 \delta_a{}^b \delta^l{}_m & 0 & 0 & 0 & 0 \end{pmatrix} \partial_l \begin{pmatrix} \tilde{E}^m_b \\ \mathcal{A}^b_m \\ \lambda \\ \lambda_m \\ \lambda_b \end{pmatrix}. \tag{B.37}$$

Clearly, the solution $(\tilde{E}^i_a, \mathcal{A}^a_i, \lambda, \lambda_i, \lambda_a) = (\tilde{E}^i_a, \mathcal{A}^a_i, 0, 0, 0)$ represents the original solution of the Ashtekar system. If the λs decay to zero after the evolution, then the solution also describes the original solution of the Ashtekar system in that stage. Since the dynamical system of $u^{(DL)}_\alpha$, (B.37), constitutes a symmetric hyperbolic form, the solutions to the λ-system are unique. Therefore, the dynamical system, (B.37), is useful for stabilizing numerical simulations from the point that it recovers the constraint surface automatically.

B.2.3 Adjusted Ashtekar system

We also try to compare a set of evolution system, which we proposed as "adjusted-system". The fundamental equations that we will demonstrate are the same with (B.27) and (B.28), but here the real-valued constant multipliers κs are not necessary equals to unity. We set $\kappa \equiv \kappa_1 = \kappa_2 = \kappa_3$ for simplicity. Apparently the set of (B.27) and (B.28) becomes the original weakly hyperbolic system if $\kappa = 0$, becomes the symmetric hyperbolic system if $\kappa = 1$ and $N = const.$. The set remains strongly hyperbolic systems for other choices of κ except $\kappa = 1/2$ which only forms weakly hyperbolic system.

B.3 Comparing numerical performance

B.3.1 Model and Numerical method

The model we present here is gravitational wave propagation in a planar spacetime under periodic boundary condition. We performed a full numerical simulation using Ashtekar's variables. We prepare two +-mode strong pulse waves initially by solving the ADM Hamiltonian constraint equation, using York-O'Murchadha's conformal approach. Then we transform the initial Cauchy data (3-metric and extrinsic curvature) into the connection variables, $(\tilde{E}^i_a, \mathcal{A}^a_i)$, and evolve them using the dynamical equations. For the presentation in this article, we apply the geodesic slicing condition (ADM lapse $N = 1$ or densitized lapse $\underset{\sim}{N} = 1$, with zero shift and zero triad lapse). We have used both the Brailovskaya integration scheme, which is a second order predictor-corrector method, and the so-called iterative Crank-Nicholson integration scheme for numerical time evolution. The details of the numerical method are described in [77].

More specifically, we set our initial guess 3-metric as

$$\hat{\gamma}_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ sym. & 1 + K(e^{-(x-L)^2} + e^{-(x+L)^2}) & 0 \\ sym. & sym. & 1 - K(e^{-(x-L)^2} + e^{-(x+L)^2}) \end{pmatrix}, \tag{B.38}$$

in the periodically bounded region $x = [-5, +5]$. Here K and L are constants and we set $K = 0.3$ and $L = 2.5$ for the plots.

In order to show the expected "stabilization behavior" clearly, we artificially add an error in the middle of the time evolution. We added an artificial inconsistent rescaling once at time $t = 6$ for the $\mathcal{A}^2_y$ component as $\mathcal{A}^2_y \to \mathcal{A}^2_y(1 + \text{error})$.

B.3.2 Differences between three levels of hyperbolicity

We have performed comparisons of stability and/or accuracy between weakly and strongly hyperbolic systems, and between weakly and symmetric hyperbolic systems[77]. (We can not compare strongly and symmetric hyperbolic systems directly, because these two requires different gauge conditions.)

We omit figures in this report, but one can see a part of results in Fig.9 and Fig.10. We may conclude that higher level hyperbolic system gives us slightly accurate evolution. However, if we evaluate the magnitude of L2 norms, then we also conclude that there is no measurable differences between strongly and symmetric hyperbolicities. This last fact will be supported more affirmatively in the next experiments.

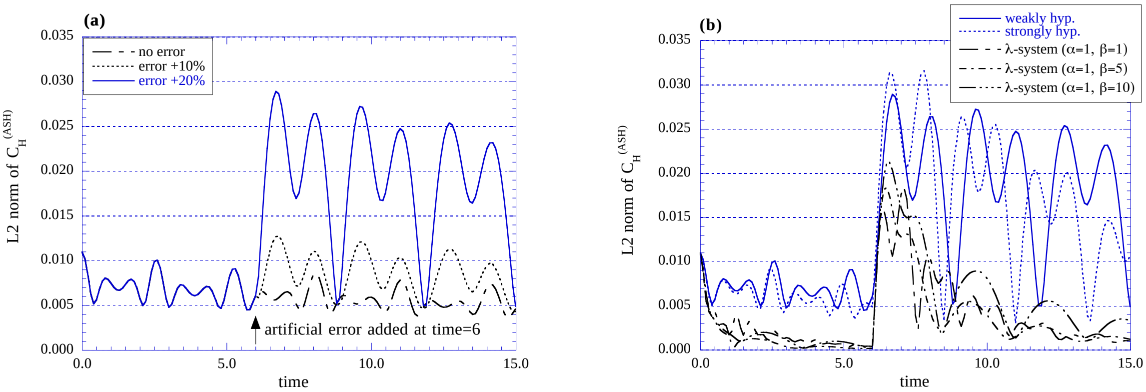

Figure 9: Demonstration of the Ashtekar-λ-system for the cases of plane wave propagation under the periodic boundary. We plot the L2 norm of the Hamiltonian constraint equation, $\mathcal{C}_H$. Fig. (a) shows how the system goes bad depending on the amplitude of artificial error at $t = 6$. All the lines are of the evolution of Ashtekar's original equation (no λ-system). Fig. (b) shows the effect of λ-system. All the lines are 20% error amplitude, but shows the difference of evolution equations. The solid line is for Ashtekar's original equation (the same as in Fig.(a)), the dotted line is for the strongly hyperbolic Ashtekar's equation. Other lines are of λ-systems, which produces better performance than that of the strongly hyperbolic system. (Reprinted from [94], ©IOP 2001)

B.3.3 Demonstrating "λ-system"

Next, we show a result of the "λ-system" [94]. Fig.9 (a) shows how the violation of the Hamiltonian constraint equation, $\mathcal{C}_H$, become worse depending on the term error. The oscillation of the L2 norm $\mathcal{C}_H$ in the figure due to the pulse waves collide periodically in the numerical region. We, then, fix the error term as a 20% spike, and try to evolve the same data in different equations of motion, i.e., the original Ashtekar's equation [solid line in Fig.9 (b)], strongly hyperbolic version of Ashtekar's equation (dotted line) and the above λ-system equation (other lines) with different βs but the same α. As we expected, all the λ-system cases result in reducing the Hamiltonian constraint errors.

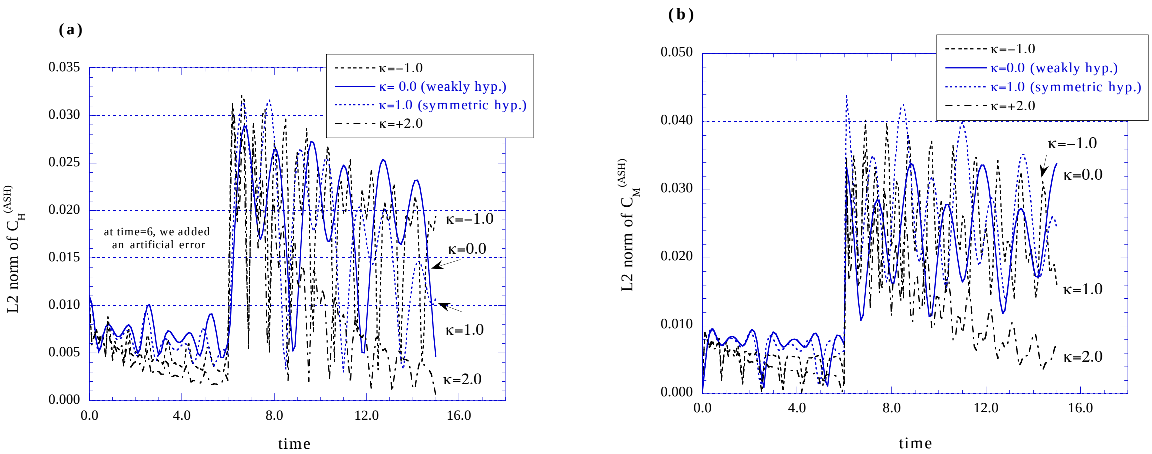

Figure 10: Demonstration of the adjusted-Ashtekar systems. The experiments as in Fig.9(b). Fig. (a) and (b) are L2 norm of the Hamiltonian constraint equation, $\mathcal{C}_H$, and momentum constraint equation, $\mathcal{C}_{Mx}$, respectively. The solid line is the case of $\kappa = 0$, that is the case of "no adjusted" original Ashtekar equation (weakly hyperbolic system). The dotted line is for $\kappa = 1$, equivalent to the symmetric hyperbolic system. We see other line ($\kappa = 2.0$) shows better performance than the symmetric hyperbolic case. (Reprinted from [94], ©IOP 2001)

B.3.4 Demonstrating "adjusted system"

We here examine how the adjusted multipliers contribute to the system's stability [94]. In Fig.10, we show the results of this experiment. We plot the violation of the constraint equations both $\mathcal{C}_H$ and $\mathcal{C}_{Mx}$. An artificial error term was added in the same way as above. The solid line is the case of $\kappa = 0$, that is the case of "no adjusted" original Ashtekar equation (weakly hyperbolic system). The dotted line is for $\kappa = 1$, equivalent to the symmetric hyperbolic system. We see other line ($\kappa = 2.0$) shows better performance than the symmetric hyperbolic case.

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